The p-adic Simpson correspondence for projective varieties – sketch of proof

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Abstract

We give an introduction to p-adic non-abelian Hodge theory and explain the statement of the p-adic Simpson correspondence for smooth projective varieties. We then sketch the proof, following [25] and focusing on the main ideas and their motivation.

These are lecture notes for a minicourse on the p-adic Simpson correspondence given at the Brin Mathematics Research Center at the University of Maryland in November 2025.

1 Introduction to the p-adic Simpson correspondence

This note aims to give an introduction to the p-adic Simpson correspondence with an emphasis on the comparison to complex geometry. This topic is currently very active, and this note is not conceived as a survey article, or an overview of the broader area of p-adic non-abelian Hodge theory. Rather, we focus on the motivation and statement of the p-adic Simpson correspondence for smooth projective varieties, as well as the main ideas of its proof.

The goal of this first section is to explain from a purely algebraic-geometric perspective what Hodge theory and non-abelian Hodge theory are about, and to explain by analogy the goals of p-adic Hodge theory and p-adic non-abelian Hodge theory.

1.1 Hodge theory

We begin in complex geometry: Let X be a smooth projective variety over $\mathbb{C}$.

Hodge theory is about comparing different cohomology theories of X: In this context, the three classical cohomology theories associated to X that we are interested in are the following.

1. **Betti cohomology**:
$$H^n_B(X, \mathbb{C})$$
Here we consider $X(\mathbb{C})$ as a topological space and consider its singular cohomology.
2. **de Rham cohomology**:
$$H^n_{\mathrm{dR}}(X)$$
Here we consider X as a differentiable manifold and consider the cohomology of its de Rham complex.
3. **Hodge cohomology**:
$$H^n_{\mathrm{Hdg}}(X, \mathbb{C}) := \bigoplus_{i+j=n} H^j(X, \Omega^i_X).$$
Here we consider X as an algebraic variety and make use of coherent cohomology.

Theorem 1.1 (de Rham, Hodge, 1930s)**.** *There are canonical and functorial isomorphisms*

$$H^*_B(X,\mathbb{C}) = H^*_{\mathrm{dR}}(X,\mathbb{C}) = H^*_{\mathrm{Hdg}}(X,\mathbb{C}).$$

The first is the de Rham comparison isomorphism, the second is called Hodge decomposition.

1.2 non-abelian Hodge theory

Non-abelian Hodge theory is about generalising results from Hodge theory to cohomology with coefficients. In particular, it is about comparing categories of coefficients for these cohomology theories:

1. Betti cohomology can be formed with coefficients in any $\mathbb{C}$-local system $\mathbb{L}$ on $X(\mathbb{C})$:
$$H^*_B(X,\mathbb{L}).$$
This can be computed as the sheaf cohomology of $\mathbb{L}$ on the topological space $X(\mathbb{C})$.
2. de Rham cohomology can be considered with coefficients in any vector bundle with flat connection (E,∇)
$$H^*_{\mathrm{dR}}(X,(E,\nabla))$$
defined as the cohomology of the de Rham complex associated to (E,∇).
3. Hodge cohomology can be defined for any **Higgs bundle** (E,θ): This is the datum of a vector bundle E on X together with an $\mathcal{O}_X$-linear homomorphism $\theta : E \to E\otimes\Omega^1$ satisfying a commutativity condition: This commutativity condition can be expressed by saying that θ induces a homomorphism of $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}\Omega_X^\vee \to \mathrm{End}(E)$. The cohomology of a Higgs bundle (E,θ), the called "Dolbeault cohomology"
$$H^*_{\mathrm{Dol}}(X,(E,\theta))$$
is then defined as the hypercohomology of the Dolbeault complex, which is defined analogously to the algebraic de Rham complex.

All three categories turn out to be naturally equivalent: This is based on the work of many people, culminating in two articles of Carlos Simpson in the 1990s:

Theorem 1.2 ([37],[38])**.** *1. There are natural equivalences of categories*

$$\left\{\begin{matrix}\mathbb{C}\textit{-local systems}\\ \textit{on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{vector bundle with}\\ \textit{flat connection on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Higgs bundles on } X\\ \textit{semistable with } c_i=0\end{matrix}\right\}$$

The first is the classical Riemann–Hilbert correspondence, the second is the Corlette–Simpson correspondence [37]. Here "$c_i = 0$" means "vanishing $\mathbb{Q}$-Chern classes".

2. For any objects corresponding to each other under the equivalences in (i)

$$\mathcal{L} \longleftrightarrow (V,\nabla) \longleftrightarrow (E,\theta),$$

there are natural isomorphisms of their respective cohomologies

$$H^*_B(X,\mathcal{L}) = H^*_{\mathrm{dR}}(X,(V,\nabla)) = H^*_{\mathrm{Dol}}(X,(E,\theta))$$

Simpson moreover proved in [38] that all three categories admit natural quasi-projective coarse moduli spaces of rank n objects. For Betti and de Rham cohomology, the respective moduli spaces are isomorphic. In contrast, Simpson shows that the de Rham and Higgs moduli spaces are only real-analytically isomorphic. They are in general not isomorphic as complex varieties, and in particular not as algebraic varieties (even for rank $n = 1$ and X a curve).

The goal of this course is to explain that there is a very similar story of all of the above in p-adic geometry!

In order to explain this, we begin with the analogue of Hodge theory in p-adic geometry.

1.3 p-adic Hodge theory

Let p be a prime and let X be a smooth projective variety over $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$, considered as an adic space $\mathrm{Spa}(\mathbb{C}_p)$. Originally, p-adic Hodge theory is about comparing different p-adic cohomology theories for X (hence the name of this subject). This is a vast topic, one reason being that there are in fact many meaningful p-adic cohomology theories that one could consider. In this talk, we focus exclusively on the "Hodge–Tate" part of the story: This means that we compare the following three cohomology theories associated to X:

1. **Hodge cohomology:** Since Hodge cohomology is completely algebraic, this can be defined in exactly the same way as in the complex theory:

$$H^*_{\mathrm{Hdg}}(X) = \bigoplus_{i+j=*} H^j(X, \Omega^i) \quad \text{as before}$$

   That being said, especially when we consider Galois actions, it is better in the p-adic setting to replace Ω^i by $\Omega^i(-i) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)^{\otimes i}, \Omega^i)$ where $\mathbb{Z}_p(1) = \varprojlim_k \mu_{p^k}(\mathbb{C}_p)$. We can ignore this "Tate twist" by choosing a compatible system of p-th unit roots in $\mathbb{C}_p$.

2. **p-adic étale cohomology** as defined by Grothendieck:

$$H^*_{\text{ét}}(X, \mathbb{C}_p) := \left(\varprojlim_n H^*_{\text{ét}}(X, \mathbb{Z}/p^k) \right) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$$

3. **v-cohomology** as defined by Scholze [36][33]: For this we consider X as an adic space over $\mathbb{C}_p$ and form the v-site of all perfectoid spaces over X:

$$X_v = \mathrm{Perf}_X \quad \text{equipped with the } v\text{-topology}$$

   This carries a natural structure sheaf $\mathcal{O}$ sending $T \mapsto H^*(T, \mathcal{O}_T)$, making X_v into a ringed site. One then simply defines

$$H^*_v(X, \mathcal{O})$$

   as the internal cohomology of the ringed site $(X_v, \mathcal{O})$.

Once again, it turns out that these three cohomologies can be compared to each other. Once again, this result is due to the contributions of many mathematicians over an extensive period of time, starting with Tate's famous article about p-divisible groups [39] from 1967:

Theorem 1.3 (Tate [39], Raynaud, Fontaine, ..., Faltings [13], Scholze [34])**.** *There are natural isomorphisms*

$$H^*_{\text{ét}}(X, \mathbb{C}_p) = H^*_v(X, \mathcal{O}) = H^*_{\mathrm{Hdg}}(X).$$

The isomorphism between the first and third term is called Hodge–Tate decomposition. The comparison between the first and second is the Primitive Comparison Theorem of [34].

The second isomorphism is actually not completely canonical: Rather, it depends on the choice of a $B_2 := B^+_{\mathrm{dR}}/\xi^2$-lift of X. There is a canonical choice of such a lift if we are given a model X_0 of X over $\overline{\mathbb{Q}}_p$.

The Hodge–Tate decomposition of Theorem 1.3 may be regarded as a p-adic analogue of the Hodge decomposition of Theorem 1.1. While it was already conjectured in this generality by Tate (hence the name), it was ultimately proved by Faltings, and later by Scholze. In both cases, the comparison uses v-cohomology (or, in Faltings' case, a slightly different site which for these purposes can be used to a similar effect): Even though the statement of the Hodge–Tate decomposition only contains more classical objects like étale and coherent cohomology, it is therefore reasonable to say that its proof goes through v-cohomology as a natural intermediary.

1.4 p-adic non-abelian Hodge theory

In analogy to the complex story, p-adic non-abelian Hodge theory is about generalizing comparison theorems of p-adic Hodge theory to coefficient objects. The natural coefficient objects for the three p-adic cohomology theories mentioned above are:

1. **Hodge cohomology:** Here we take Higgs bundles, as in the complex theory. Their definition is purely algebraic, so we can make the same definition over $\mathbb{C}_p$. There is only one small subtlety: Once again, it turns out to be more natural to work with $\Omega_X(-1)$ instead of Ω_X in the definition. This difference can be ignored after choosing compatible unit roots in $\mathbb{C}_p$.
2. **p-adic étale cohomology:** For this we use $\mathbb{C}_p$-local systems, in analogy to Betti cohomology over $\mathbb{C}$. These are defined as the isogeny category of $\mathcal{O}_{\mathbb{C}_p}$-local systems, and there are several ways to define the latter. Let us first give a classical definition, and then explain a more modern one which will give a further indication for why v-covers are useful in this context:

   Definition 1.4. An $\mathcal{O}_{\mathbb{C}_p}$-local system on $X_{\text{ét}}$ is an inverse system $(\mathbb{L}_k)_{k\in\mathbb{N}}$ of $\mathcal{O}_{\mathbb{C}_p}/p^k$-local systems on the usual étale site $X_{\text{ét}}$ with isomorphisms $\mathbb{L}_{k+1}/p^k \cong \mathbb{L}_k$ for all k.

   For example, for any $\mathbb{Z}_p$-local system $\mathbb{L}$, we obtain a $\mathbb{C}_p$-local system $\mathbb{L}\otimes\mathbb{C}_p$ defined by tensoring with $\mathcal{O}_{\mathbb{C}_p}$ and inverting p.

   Definition 1.5. An $\mathcal{O}_{\mathbb{C}_p}$-local system on X_v is a sheaf on X_v which is locally isomorphic to $\underline{\mathcal{O}_{\mathbb{C}_p}^n}$ where

   $$\underline{\mathcal{O}_{\mathbb{C}_p}} := \varprojlim \mathcal{O}_{\mathbb{C}_p}/p^k$$

   is a pro-locally constant sheaf on X_v.

   Proposition 1.6. *Definition 1.4 and Definition 1.5 define equivalent categories.*

   To explain the proof, recall that over $\mathbb{C}$, any local system becomes trivial on the topological universal cover. There is an analogue of this over $\mathbb{C}_p$: the **diamantine universal cover**

   $$\widetilde{X} := \varprojlim_{X'\to X} X'.$$

   This limit is to be taken in Scholze's category of diamonds over X. For smooth projective curves of genus ≥ 1 or for abelian varieties over $\mathbb{C}_p$, the diamond $\widetilde{X}$ is in fact represented by a perfectoid space [9][8]. The space $\widetilde{X}\to X$ is a very useful example of an object in X_v. In fact, it is a $\pi_1^{\text{ét}}(X)$-torsor in X_v, essentially by definition of the étale fundamental group $\pi_1^{\text{ét}}(X)$, which is a profinite group.

   Proposition 1.7. *There is an equivalence of categories*

   $$\mathrm{Rep}_{\mathbb{C}_p}(\pi_1^{\text{ét}}(X)) := \left\{\begin{matrix}\textit{continuous representations of } \pi_1^{\text{ét}}(X)\\ \textit{on finite dimensional } \mathbb{C}_p\textit{-vector spaces}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\mathbb{C}_p\textit{-local systems on } X_v\\ \textit{(as in Definition 1.5)}\end{matrix}\right\}$$

   $$\pi_1(X)\to \mathrm{GL}(M) \quad\longmapsto\quad \widetilde{X}\times M/\pi_1^{\text{ét}}(X)$$

   This also explains Proposition 1.6 because $\mathrm{Rep}_{\mathbb{C}_p}(\pi_1^{\text{ét}}(X))$ is clearly the isogeny category of continuous $\mathcal{O}_{\mathbb{C}_p}$-representations of $\pi_1(X)$. Since these have finite image mod p^k for any k, these are in turn equivalent to $\mathcal{O}_{\mathbb{C}_p}$-local systems in the sense of Definition 1.4.

   This finishes the discussion of coefficients for p-adic étale cohomology: The upshot is that already in this more classical setting, the perfectoid perspective provided by X_v turns out to be helpful.

3. **Coefficients for v-cohomology:**

Our category of coefficients for cohomology on X_v will simply be **v-vector bundles**. Once again, like for étale cohomology, there are two slightly different definitions of this:

1. Faltings' "generalized representations", which are a coherent analogue of the classical definition of $\mathbb{C}_p$-local systems on $X_{\text{ét}}$ (Definition 1.4).
2. finite locally free $\mathcal{O}$-modules on X_v, which are a coherent analogue of the v-topological definition of $\mathbb{C}_p$-local systems on X_v (Definition 1.5).

Once again, both categories turn out to be equivalent [22, Proposition 2.3]. In the following, we shall work with the latter category.

A natural question in this context is how v-vector bundles can be compared to algebraic or analytic vector bundles on X: To explain this, we recall that there are natural morphisms of topoi

$$X_{\text{an}} \leftarrow X_{\text{ét}} \leftarrow X_v$$

For smooth rigid spaces, these induce natural fully faithful functors

$$\left\{\begin{matrix}\text{vector bundles}\\ \text{on } X_{\text{Zar}}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{vector bundles}\\ \text{on } X_{\text{an}}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{vector bundles}\\ \text{on } X_{\text{ét}}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{vector bundles}\\ \text{on } X_v\end{matrix}\right\}.$$

The first two of these are equivalence of categories by rigid GAGA [28] and rigid étale descent [10]. In contrast, the third is usually not essentially surjective. (See [23, §1] for a more detailed discussion of vector bundles also in various other p-adic analytic topologies).

To get a better understanding of v-vector bundles, we first compare them to local systems:

Proposition 1.8 ([23, Theorem 5.2])**.** *i) There is a natural fully faithful functor*

$$\begin{aligned}\left\{\begin{matrix}\mathbb{C}_p\textit{-local systems}\\ \textit{on } X_v\end{matrix}\right\} &\hookrightarrow \left\{\begin{matrix}\textit{v-vector bundles}\\ \textit{on } X_v\end{matrix}\right\}\\ \mathcal{L} &\mapsto \mathcal{L}\otimes_{\mathbb{C}_p}\mathcal{O} := V\end{aligned}$$

ii) For any $\mathbb{C}_p$-local system $\mathcal{L}$ and $V := S(\mathcal{L})$, there is a canonical isomorphism

$$H^*_{\text{ét}}(X,\mathcal{L}) \;=\; H^*_v(X,\mathcal{V})$$

This explains the comparison between coefficients for p-adic étale cohomology and v-vector bundles. We note that from the concrete definition, this functor looks like a p-adic equivalent of the Riemann–Hilbert functor. In fact, this analogy can be made more precise: Let us mention in this context the work of Liu–Zhu [29], Min–Gao–Wang [17] and the upcoming work of Bhatt–Lurie [5].

1.5 The p-adic Simpson correspondence

It remains to compare coefficients between v-cohomology and Hodge cohomology, namely v-vector bundles and Higgs bundles. This is the content of the p-adic Simpson correspondence, which is the main result of this lecture series. Once again, this is the result of the work of many authors, starting around 2005 with articles of Deninger–Werner [11] and Faltings [14] (who coined the name "p-adic Simpson"), and developed by Abbes–Gros–Tsuji [2][1][40], Liu–Zhu [29], Wang [41] and many others.

In these notes we shall focus on the following projective version, because it is closest in its formulation to the complex Corlette–Simpson correspondence:

Theorem 1.9 (p-adic Simpson correspondence, Faltings [14], H. [25, Theorems 5.1 and 5.5])**.** *Let X be a smooth projective variety over $\mathbb{C}_p$.*

1. *There is a natural equivalence of categories*

$$S : \{v\text{-vector bundles on } X\} \longleftrightarrow \{\text{Higgs bundles on } X\}.$$

   It is a rank-preserving exact tensor functor.

2. *Given any v-vector bundle V and $(E, \theta) := S(V)$, there is a canonical isomorphism*

$$H^*_v(X, V) \;=\; H^*_{\mathrm{Dol}}(X, (E, \theta)).$$

Some comments about this Theorem:

1. The case $V = \mathcal{O}$ of part 2 recovers the Hodge decomposition

$$H^n_v(X, \mathcal{O}) = \bigoplus_{i+j=n} H^i(X, \Omega^j(-j)).$$

   Hence the p-adic Simpson correspondence generalises the Hodge–Tate decomposition by generalising it to coefficients. We learnt this perspective from Abbes–Gros–Tsuji [2].

2. The theorem holds more generally for any smooth proper rigid space over any algebraically closed complete extension of $\mathbb{Q}_p$. In this note we focus on the projective case to simplify both the exposition and the comparison to the complex story.

3. The functor S is not canonical, rather it depends on two choices:
   i) A lift $\mathbb{X}$ of X along the square-zero thickening $B_2 \to \mathbb{C}_p$. Such a lift always exists (by Conrad–Gabber spreading-out, see [19]). If X comes from a model X_0 over $\overline{\mathbb{Q}}_p$, there is a canonical such choice induced by the natural map $\overline{\mathbb{Q}}_p \to B_2$.
   ii) A p-adic exponential $\mathrm{Exp} : \mathbb{C}_p \to 1 + \mathfrak{m}_{\mathbb{C}_p}$. This is defined as a splitting of the p-adic logarithm $\log : 1 + \mathfrak{m}_{\mathbb{C}_p} \to \mathbb{C}_p$ by a continuous homomorphism. This always exists as well, but this time there is no canonical choice.

   Both choices are really necessary, and outside of trivial cases, one can in fact recover both $\mathbb{X}$ and Exp from any given correspondence S. The "naturality" of S means that the correspondence is compatible with respect to pullbacks that lift to the given B_2-lifts.

4. Combined with Proposition 1.8, Theorem 3.1 yields a natural fully faithful embedding

$$\mathrm{Rep}_{\mathbb{C}_p}(\pi_1(X)) \;\hookrightarrow\; \{\text{Higgs bundles on } X\}$$

   which underlines the analogy to the complex Corlette–Simpson correspondence. It is then natural to ask whether the essential image can be described like in complex geometry:

   Question 1.10 (Faltings)**.** What is the essential image of this functor? Is it semistable Higgs bundles with vanishing $\mathbb{Q}$-Chern classes?

   This is thought to be a really hard question. The answer is known only in certain special cases, such as line bundles and abelian varieties, and partially in special cases such as Higgs field 0 due to Deninger–Werner [11][12] and Würthen [42].

   Like for the Hodge–Tate decomposition, the above faithful functor is between more classical objects, but its construction goes through v-vector bundles as a natural intermediary.

1.6 The local p-adic Simpson correspondence

There is also a local version of the p-adic Simpson correspondence for non-proper varieties. In fact, in contrast to the complex story, this was understood before the proper case, in extensive works of Faltings [14][15], Abbes–Gros [2][1], Tsuji [2][40] and Wang [41]:

Theorem 1.11 (Local correspondence)**.** *Let X be any smooth rigid space with a given lift of X to B_2. Then there is an exact tensor equivalence of categories*

$$\{\textit{small v-vector bundle}\} \longleftrightarrow \{\textit{small Higgs bundle.}\}$$

In fact, Faltings' strategy in [14] to prove Theorem 1.9 is to deduce the proper correspondence in the case of curves from the above local correspondence by descent [14].

In comparison to Theorem 1.9, the local Theorem 1.11 does not require X to be proper, and we can even drop the choice of exponential. The price we pay is that we only get a correspondence for "small" objects, which roughly means "p-adically close to the trivial bundle". More precisely, for Higgs fields, smallness is a certain topological convergence condition for the powers of θ.

One also has a comparison of cohomologies in this case. In fact, more generally, there is also a derived version of this small correspondence [3]. From this one recovers the cohomological comparison by considering $\mathrm{RHom}(\mathcal{O}, -)$.

1.7 Summary of Talk I

The following diagram summarises how the comparisons between categories of coefficients for cohomology theories discussed in this section fit together (the reader is invited to choose an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ to make everything fit into one picture):

$$\begin{tikzcd}
& \mathbb{C} - \text{analytic} & \text{algebraic} & \mathbb{C}_p - \text{analytic} & \\
& \left\{\begin{matrix}\text{vector bundles}\\ \text{flat connection}\end{matrix}\right\} \arrow[r, leftrightarrow, "\substack{\text{Corlette}\\ \text{-Simpson}}"] \arrow[d, leftrightarrow, "\text{Riemann–Hilbert}"'] & \left\{\begin{matrix}\text{Higgs}\\ \text{bundles}\end{matrix}\right\} \arrow[r, leftrightarrow, "\substack{p\text{-adic}\\ \text{Simpson}}"] & \left\{\begin{matrix}\text{v-vector}\\ \text{bundles}\end{matrix}\right\} \arrow[d, leftrightarrow, "p\text{-adic Riemann-Hilbert}"] & \\
\mathrm{Rep}_{\mathbb{C}}(\pi_1(X)) \arrow[r, leftrightarrow, "\widetilde{X}"'] & \left\{\begin{matrix}\mathbb{C}\text{-local}\\ \text{systems}\end{matrix}\right\} & & \left\{\begin{matrix}\mathbb{C}_p\text{-local}\\ \text{systems}\end{matrix}\right\} \arrow[r, leftrightarrow, "\widetilde{X}"'] & \mathrm{Rep}_{\mathbb{C}_p}(\pi_1^{\text{ét}}(X))
\end{tikzcd}$$

2 The case of rank one

In this section, we explain how to prove the p-adic Simpson correspondence in an easier special case, namely for line bundles: Following [23], we will sketch the proof of the following result:

Theorem 2.1. *Let X be any smooth proper rigid space over $\mathbb{C}_p$. Then choices of a B_2-lift $\mathbb{X}$ of X and of an exponential* $\mathrm{Exp} : \mathbb{C}_p \to 1 + m_{\mathbb{C}_p}$ *induce a natural equivalence of categories*

$$S_1 : \{v\text{-line bundles on } X\} \longleftrightarrow \{\text{Higgs line bundles on } X\}.$$

This case turns out to be interesting and instructive: On the one hand, it is amenable to explicit cohomological computation. On the other hand, several key ideas for the general case are already visible in this context, for example it explains how lift and exponential get used.

From now on, we shall for simplicity ignore Tate twists by choosing an isomorphism $\mathbb{Z}_p(1) = \mathbb{Z}_p$, i.e. a compatible system of unit roots in $K := \mathbb{C}_p$.

2.1 Recollection on the Hodge–Tate sequence

We begin by noting that the isomorphism classes of objects on both sides of S are given by the cohomology groups

$$H^1_v(X, \mathcal{O}^\times) \quad \text{and} \quad H^1_{\mathrm{an}}(X, \mathcal{O}^\times) \times H^0(X, \Omega^1).$$

Our starting point is the observation that the envisioned comparison isomorphism between these groups looks like a "multiplicative version" of the Hodge–Tate decomposition (Theorem 1.3)

$$H^1_v(X, \mathcal{O}) = H^1_{\mathrm{an}}(X, \mathcal{O}) \oplus H^0(X, \Omega^1),$$

in which "$\mathcal{O}$ gets replaced by $\mathcal{O}^\times$". We will therefore take inspiration from Scholze's proof of the Hodge-Tate spectral sequence [34][35], based on the earlier work of Faltings [13]. Let us recall the main steps of its proof.

The idea is to consider the morphism of topoi

$$v : X_v \to X_{\bar{\eta}}$$

and its Leray spectral sequence:

Theorem 2.2. *(Scholze [34][35], Bhatt-Morrow-Scholze [6]):*

1. *For any $j \in \mathbb{N}$, there is a canonical isomorphism*

$$R^j \nu_* \mathcal{O}_X = \Omega^j_X.$$

2. *The Leray sequence*

$$E_2^{i,j} = H^i(X, \Omega^j_X) \Rightarrow H^{i+j}_v(X, \mathcal{O})$$

degenerates at the E_2-page.

Looking at the associated 3-term exact sequence of low degrees, this yields a short exact sequence

$$0 \to H^1(X, \mathcal{O}) \to H^1_v(X, \mathcal{O}) \xrightarrow{\mathrm{HT}} H^0(X, \Omega^1_X) \to 0.$$

The datum of a B_2-lift of X to B_2 induces a splitting of this sequence: More precisely, it already induces a splitting in the derived category (see [19, Theorem 7.4.9])

$$R\nu_* \mathcal{O} = \bigoplus_{i=0}^{d} \Omega^d_X[-d].$$

2.2 The multiplicative Hodge–Tate sequence

It turns out that there is indeed a direct analogue of all results of the last subsection for $\mathcal{O}^\times$ instead of $\mathcal{O}$, which we call the "multiplicative Hodge–Tate sequence":

Theorem 2.3. *(H. [21, Corollary 2.19], Gerth [18, Theorem 3.26])*

1. *For the morphism $\nu : X_v \to X_{\text{ét}}$, there is for any $j \geq 0$ a canonical isomorphism*

$$R^j\nu_*\mathcal{O}_X^\times = \begin{cases} \mathcal{O}_X & j = 0, \\ \Omega_X^j & j \geq 1 \end{cases}$$

2. *The Leray sequence*

$$E_2^{i,j} = \begin{cases} H^i_{\text{ét}}(X, \Omega^j) & j \geq 1 \\ H^i_{\text{ét}}(X, \mathcal{O}^\times) & j = 0 \end{cases} \quad \Rightarrow \quad H^{i+j}_{\text{ét}}(X, \mathcal{O}^\times)$$

degenerates at the E_2-page.

Once again, we deduce in low degrees:

Corollary 2.4. *There is a short exact sequence*

$$0 \longrightarrow H^1_{\text{an}}(X, \mathcal{O}^\times) \longrightarrow H^1_v(X, \mathcal{O}^\times) \xrightarrow{\text{HTlog}} H^0(X, \Omega^1) \longrightarrow 0$$

This already comes very close to giving the desired comparison of isomorphism classes: The term in the middle classifies v-line bundles, whereas the direct sum of the outer terms classifies Higgs line bundles!

2.3 Sketch of the proof of Theorem 2.3.1

For notational convenience, let us set $K := \mathbb{C}_p$. We consider the subgroup $1 + m_K\mathcal{O}^+ \subseteq \mathcal{O}^\times$. This sits in short exact sequences of sheaves on $X_{\text{ét}}$.

$$1 \longrightarrow 1 + m_K\mathcal{O}^+ \longrightarrow \mathcal{O}^\times \longrightarrow \overline{\mathcal{O}}^\times \longrightarrow 1$$

where $\overline{\mathcal{O}}^\times := \mathcal{O}^\times/(1 + m_K\mathcal{O}^+)$. The first term in turn sits in a short exact sequence

$$1 \longrightarrow \mu_{p^\infty} \longrightarrow 1 + m_K\mathcal{O}^+ \xrightarrow{\log} \mathcal{O} \longrightarrow 1$$

of sheaves on $X_{\text{ét}}$. This sequence is one of the reasons why in this context, we prefer to work with $X_{\text{ét}}$ instead of X_{an}. Another is the following property:

Lemma 2.5 ([23, Lemma 2.17], [21, Proposition 4.1.])**.** *The sheaf $\overline{\mathcal{O}}^\times$ "commutes with limits" in X_v. It follows from this that*

$$\nu_*\overline{\mathcal{O}}^\times = \overline{\mathcal{O}}^\times$$
$$R^i\nu_*\overline{\mathcal{O}}^\times = 1 \quad \textit{for } i \geq 1.$$

The logarithm now induces isomorphism for any $i \geq 1$:

$$R^i\nu_*\mathcal{O}^\times \cong R^i\nu_*(1 + m_K\mathcal{O}^+) \xrightarrow{\log} R^i\nu_*\mathcal{O} \cong \Omega^i.$$

where the last isomorphism comes from Theorem 2.2. All in all, this shows Theorem 2.3.1.

2.4 Sketch of the proof of Corollary 2.4

In rank one, the fully faithful functor

$$\mathrm{Rep}_{\mathbb{C}_p}(\pi_1(X)) \hookrightarrow \{\text{v-vector bundles}\}$$

from Question 1.10 yields a commutative diagram

$$\begin{tikzcd}
 & H^1_v(X, \mathcal{O}^\times) \arrow[r, "\mathrm{HTlog}"] & H^0(X, \Omega^1) \\
\mathrm{Hom}_{\mathrm{cts}}(\pi_1(X), 1 + m_{\mathbb{C}_p}) \arrow[r] \arrow[d, "\log"] & H^1_v(X, 1 + m_{\mathbb{C}_p}\mathcal{O}^+) \arrow[u] \arrow[r, "\mathrm{HTlog}"] \arrow[d, "\log"] & H^0(X, \Omega^1) \arrow[u, equal] \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{cts}}(\pi_1(X), \mathbb{C}_p) \arrow[r, "\cong"] & H^1_v(X, \mathcal{O}) \arrow[r, "\mathrm{HT}"] & H^0(X, \Omega^1)
\end{tikzcd}$$

This also illustrates why HTlog got its name.

Now we can explain the role of the choices in the statement of the p-adic Simpson correspondence:

1. The datum of a B_2-lift splits the morphism HT on the bottom line.
2. The datum of an exponential induces a splitting of the map log on the bottom left.

In combination, these two splitting yield a splitting of the middle row, and thus of the top row. All in all, this defines a splitting of the multiplicative Hodge–Tate sequence, Corollary 2.4:

$$H^1_\nu(X, \mathcal{O}^\times) = H^1_{\text{ét}}(X, \mathcal{O}^\times) \times H^0(X, \Omega^1).$$

This explains the definition of the p-adic Simpson correspondence for line bundles, at least on the level of isomorphism classes. (We refer to [23, §4.4] for more details on the above argument.)

2.5 Moduli spaces

We now sketch a second, different proof of Corollary 2.4, using moduli spaces. This will be more work, but has the advantage to generalise: First, the moduli-theoretic argument gives the more general result Theorem 2.3.2. Second, we will use a very similar moduli-theoretic argument for proving exactness when we later consider higher dimensions.

Definition 2.6 ([20])**.** The diamantine étale Picard functor of X is the presheaf

$$\begin{aligned}
\mathrm{Pic}_X : X_v &\longrightarrow \mathrm{Ab} \\
T &\longmapsto H^1_{\text{ét}}(X \times T, \mathcal{O}^\times)/H^1_{\text{ét}}(T, \mathcal{O}^\times).
\end{aligned}$$

The **v-Picard functor** of X is the analogous presheaf describing v-line bundles

$$\begin{aligned}
\mathrm{vPic}_X : X_v &\longrightarrow \mathrm{Ab} \\
T &\longmapsto H^1_v(X \times T, \mathcal{O}^\times)/H^1_v(T, \mathcal{O}^\times).
\end{aligned}$$

Theorem 2.7 ([20, Theorem 2.7])**.** *Let X be any smooth projective variety over $\mathbb{C}_p$.*

i) Pic_X *is represented by the analytification of the usual algebraic Picard functor of X.*

ii) vPic_X *is represented by a rigid-analytic group.*

iii) There is a short exact sequence of rigid-analytic groups

$$1 \to \mathrm{Pic}_X \to \mathrm{vPic}_X \to H^0(X, \Omega^1) \otimes \mathbb{G}_a \to 0$$

It is worth pointing out that vPic_X is genuinely a rigid-analytic object even though X is algebraic, even if X is for example any smooth proper curve of genus ≥ 1.

From the last short exact sequence, we deduce the short exact sequence of Corollary 2.4 simply by taking $\mathbb{C}_p$-points. In other words, Theorem 2.7 yields a geometrization of this sequence from abelian groups to rigid group varieties. The point is that this additional geometric structure actually helps in proving the right-exactness, because it allows us to give a geometric argument, as we now explain:

A variant of Theorem 2.3.1 for X replaced by $X \times T$ for any perfectoid T induces a left-exact sequence

$$1 \to \mathrm{Pic}_X \to \mathrm{vPic}_X \to H^0(X, \Omega^1) \otimes \mathbb{G}_a$$

To see that this is right-exact, we first observe that it is right-exact after inverting p: This follows by comparing this sequence to character varieties, more precisely to the logarithm map

$$\mathrm{Hom}(\pi_1(X), \mathbb{G}_m) \xrightarrow{\log} \mathrm{Hom}(\pi_1(X), \mathbb{G}_a).$$

Since the exponential defines partial splitting of log over an open subgroup of $\mathbb{G}_a$, this shows that the sequence is exact after inverting p.

The idea is now roughly to deduce from this that the Leray spectral sequence of ν considered for $X \times T$ for any perfectoid T induces a long exact sequence of cohomology sheaves

$$1 \to \mathrm{Pic}_X \to \mathrm{vPic}_X \to H^0(X, \Omega^1) \otimes \mathbb{G}_a \xrightarrow{\partial} \underline{H^2(X, \mu_{p^\infty})}.$$

Since the third term $H^0(X, \Omega^1) \otimes \mathbb{G}_a$ and fourth term $\underline{H^2(X, \mu_{p^\infty})}$ are represented by rigid groups, the morphism ∂ has to be represented by a homomorphism of rigid groups, too. But the third term is connected while the fourth term is locally constant. Hence $\partial = 0$.

A generalisation of this kind of geometric homological argument is how Gerth proved Theorem 2.7.2.

A nice consequence of this approach via moduli spaces is that it yields a moduli-theoretic version of the p-adic Simpson correspondence, so far for line bundles:

Corollary 2.8 ([24])**.** *Let* $\mathrm{Higgs}_1 := \mathrm{Pic}_X \times H^0(X, \Omega^1) \otimes \mathbb{G}_a$ *be the moduli space of Higgs bundles of rank one, considered as a rigid space. Then* vPic_X *and* Higgs_1 *are both* Pic_X*-torsors over* $H^0(X, \Omega^1) \otimes \mathbb{G}_a$*, but the latter is* split *while the former is not.*

As it turns out, this kind of description generalises to higher rank [27].

One advantage of the moduli-theoretic approach discussed in this subsection is that it generalises in a way that we will use in the general construction of the p-adic Simpson functor.

Another advantage is that it can be used to answer Faltings' question about the essential image of representations (Question 1.10) in the special case of line bundles [24].

3 Proof of the p-adic Simpson Correspondence

In this last section, we now give a sketch of the proof of the p-adic Simpson correspondence (following [25]) for proper smooth varieties in higher rank:

Theorem 3.1. *Let X be a smooth proper rigid space. The choices of a lift $\mathbb{X}$ of X and an exponential* $\mathrm{Exp}: \mathbb{C}_p \to 1 + m_{\mathbb{C}_p}$ *lead to an equivalence of categories*

$$S : \{\textit{v-vector bundles on } X\} \longleftrightarrow \{\textit{Higgs bundles on } X\}.$$

Our starting point is the basic idea that (at least generically and for curves) one can try to reduce p-adic Simpson in higher rank to the case of line bundles of §2 by **abelianization:**

3.1 An idea that doesn't immediately work but is still helpful

Let (E, θ) be any Higgs bundle on X. We recall that by definition, this is a vector bundle on X together with an $\mathcal{O}_X$-linear homomorphism $\theta : E \to E \otimes \Omega^1$ that induces a homomorphism of $\mathcal{O}_X$-algebras

$$\theta : \mathrm{Sym}\ \Omega_X^{1\vee} \longrightarrow \mathrm{End}(E).$$

Let $B := \mathrm{im}\ \theta = \mathrm{Sym}\ \Omega_X^{1\vee}/\ker\theta$ be its image: This is an $\mathcal{O}_X$-algebra which combines the desirable properties of $\mathrm{Sym}\ \Omega_X^{1\vee}$ and $\mathrm{End}(E)$, namely it is commutative (because $\mathrm{Sym}\ \Omega_X^{1\vee}$ is) and $\mathcal{O}_X$-coherent (because $\mathrm{End}(E)$ is). Moreover, via the tautological map $B \to \mathrm{End}(E)$, this algebra acts on E.

Assume now first that X is a curve. We now follow an observation due to **Beauville–Narasimhan–Ramanan** [4]: For any "generic" (E, θ) on a curve X, the B-action makes E into an invertible B-module. Consequently, we can write (E, θ) as the pushforward

$$(E, \theta) = \pi_*(L, \tau)$$

of a Higgs line bundle (L, τ) along the finite flat cover

$$\pi : X' := \mathrm{Spa}(B) \to X.$$

For a generic Higgs bundle, this "spectral curve" X' is even a connected smooth proper curve.

With this in mind, a first idea for the proof of (the generic case in the special case of curves of) the p-adic Simpson correspondence in higher rank would be to consider the diagram

$$\begin{tikzcd}
\{\text{Higgs line bundles on } X'\} \arrow[r, "\pi_*"] \arrow[d, "S_1"] & \{\text{Higgs bundles on } X\} \arrow[d, dotted] \\
\{\text{v-line bundles on } X'\} \arrow[r, "\pi_*"] & \{\text{v-vector bundles on } X'\}
\end{tikzcd}$$

and use the p-adic Simpson correspondence for line bundles on X' from Theorem 2.1. This is a first rough idea of what "abelianization" could mean in this context: Reducing the case of GL_n to that of $\mathbb{G}_m$ on the spectral curve X'.

Problem: Unfortunately, even under the additional assumptions (X curve, X' smooth), this strategy does not immediately work: One issue is that pushforward of v-sheaves along finite flat ramified maps does not preserve v-vector bundles. Indeed, not even $\pi_*\mathcal{O}_{X'}$ is a finite flat $\mathcal{O}_X$-module on X_v in this context! The reason is that "finite flat" is not a good notion for morphisms of diamonds, as it is not preserved under pullback (this is related to Bhatt–Scholze's perfectoidization [7, §8]). In other words, we do not have base-change of coherent modules along the diagram

$$\begin{array}{ccc} X'_v & \longrightarrow & X_v \\ \downarrow & & \downarrow \\ X'_{\text{ét}} & \longrightarrow & X_{\text{ét}} \end{array}$$

Solution: The key idea is to work locally on X_v rather than on X': To explain this, recall that in classical algebraic geometry, a line bundles on $X'_{\text{ét}}$ is the same as an invertible $\pi_*\mathcal{O}_{X'_{\text{ét}}}$-module on $X_{\text{ét}}$. But for v-vector bundles, this makes a crucial difference! The latter concept is the one that we will use, as it will explain how to implement the idea of "pushforwards of line bundles". Much better, it will at the same time allow us to improve the above discussion from curves to higher dimension.

3.2 From Higgs bundles to v-vector bundles

We now explain how to construct the functor "←" from Higgs bundles to v-vector bundles in Theorem 3.1: Let (E, θ) be a Higgs bundle. Like in the last section, we consider the sheaf on $X_{\text{ét}}$

$$B := \operatorname{im}\left(\operatorname{Sym}\Omega^\vee \xrightarrow{\theta} \operatorname{End}(E)\right).$$

Let $\mathcal{B} := \nu^* B$ be the associated $\mathcal{O}_{X_v}$-algebra on X_v.

The plan will be to construct a certain invertible (i.e. locally free of rank one) $\mathcal{B}$-module $\mathcal{L}$ on X_v and define the p-adic Simpson functor as

$$S^{-1}(E, \theta) := \nu^* E \otimes_{\mathcal{B}} \mathcal{L}.$$

To construct $\mathcal{L}$, we will use moduli spaces of invertible $\mathcal{B}$-modules, for which we will generalize the v-Picard functor vPic_X from §2.

To explain this, we will in the following assume that B is locally free. The purpose of this assumption is just to simplify the exposition by avoiding some technical complications. It is harmless for curves, but the case of general coherent B is really needed in higher dimension. Making this possible is the main reason why we replace X_v by the much more restrictive proétale site $X_{\text{proét}}$ in [25]: The structure sheaf of the latter behaves as if it was "flat over $\mathcal{O}_X$" (which is definitely false for X_v), making it better suited to deal with general coherent modules.

3.3 Hodge–Tate theory for $\mathcal{B}$

Invertible $\mathcal{B}$-modules are a generalisation of v-line bundles, which are the special case of $B = \mathcal{O}_X$. In §2, we understood v-line bundles by first dealing with $\mathcal{O}$ and then passing to $\mathcal{O}^\times$. In the same vein, in order to understand invertible $\mathcal{B}$-modules, we therefore first have to understand the v-cohomology of $\mathcal{B}$. This is what we explain in this subsection, (following [25, §2]):

Proposition 3.2. *i) We have a short exact sequence*

$$0 \to H^1_{\text{ét}}(X, B) \to H^1_v(X, \mathcal{B}) \xrightarrow{\mathrm{HT}} H^0(X, B \otimes \Omega^1) \to 0$$

ii) The sequence is split by the choice of lift $\mathbb{X}$ of X.

Proof. Since B is assumed to be locally free, both follow immediately from the projection formula, which says

$$R\nu_*\nu^*B = B \otimes_{\mathcal{O}_X} R\nu_{X*}\mathcal{O}.$$

Recall that the lift $\mathbb{X}$ induces a decomposition

$$R\nu_{X*}\mathcal{O} \cong \bigoplus \Omega^i[-i]$$

$$\Rightarrow R\nu_{X*}\mathcal{B} \cong \bigoplus \Omega^i \otimes B[-i].$$

$\square$

A more conceptual way to see the splitting, which also explains how the B_2-lift enters the description, is to use the Higgs–Tate torsor of Abbes–Gros, based on the earlier work of Ogus–Vologodsky [30]: This is the sheaf on X_v defined by

$$\mathcal{L}_{\mathbb{X}} := \left\{ \text{homomorphisms } \varphi \text{ of } B_2\text{-algebras} \quad \begin{tikzcd} \nu^{-1}\mathcal{O}_{\mathbb{X}}/t^2 \arrow[r, dashed, "\varphi"] \arrow[d] & \mathbb{B}^+_{\mathrm{dR}}/t^2 \arrow[d] \\ \nu^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_X \end{tikzcd} \right\}$$

where $\mathbb{B}^+_{\mathrm{dR}}/t^2 \to \mathcal{O}$ is a square-zero thickening of the structure sheaf on X_v analogous to the map $B_2 \to \mathbb{C}_p$. By deformation theory, $\mathcal{L}_{\mathbb{X}}$ is a $\nu^*\Omega_X^\vee$-torsor on X_v (here we recall that we have chosen an isomorphism $\mathbb{Z}_p(1) = \mathbb{Z}_p$ and hence an element $t \in B_2$). For any element of $H^0(X, B \otimes \Omega^1)$, pushout along the associated map $\Omega_X^\vee \to B$ defines a splitting of HT_B:

$$H^0(X, B \otimes \Omega^1) \to H^1_v(X, \mathcal{B}).$$

In particular, the tautological class τ in $H^0(X, B \otimes \Omega^1)$ associated to the natural projection $\operatorname{Sym}\Omega^\vee \to B$ defines a canonical $\mathcal{B}$-torsor on X_v.

3.4 The multiplicative Hodge–Tate sequence for $\mathcal{B}$

We are now ready to generalise the constructions for $\mathbb{G}_m$ from §2 to $\mathcal{B}^\times$. The key to this is the definition of moduli spaces of invertible $\mathcal{B}$-modules that generalise the v-Picard variety:

We recall that we assume B to be an $\mathcal{O}_X$-coherent $\operatorname{Sym}\Omega^\vee$-algebra that is locally finite free as an $\mathcal{O}_X$-module, and $\mathcal{B} := \nu^*B$.

Definition 3.3. The Picard functor of B on X is

$$\begin{aligned} \mathrm{Pic}_B : X_v &\longrightarrow \mathrm{Ab} \\ T &\longmapsto H^1_{\text{ét}}(X \times T, B^\times)/H^1_{\text{ét}}(T, B^\times). \end{aligned}$$

The **v-Picard functor of** B on X is

$$\begin{aligned} \mathrm{vPic}_X : X_v &\longrightarrow \mathrm{Ab} \\ T &\longmapsto H^1_v(X \times T, \mathcal{B}^\times)/H^1_v(T, \mathcal{B}^\times). \end{aligned}$$

The following description of these functors generalises Theorem 2.7, which we recover as the case of $B = \mathcal{O}_X$ with trivial $\operatorname{Sym}\Omega^\vee$-action.

Theorem 3.4 ([25, Theorem 2.4]). *1. There is a natural short exact sequence of abelian sheaves on $X_{\text{proét}} \subset X_v$*

$$0 \to \mathrm{Pic}_B \to \mathrm{vPic}_B \xrightarrow{\mathrm{HTlog}} H^0(X, \Omega^1 \otimes B) \otimes \mathbb{G}_a \to 0$$

which is representable by rigid group varieties.

2. The induced sequence of Lie algebras (i.e. tangent spaces at the identity) is precisely the Hodge–Tate sequence of B:

$$0 \to H^1_{\text{ét}}(X, B) \to H^1_v(X, \mathcal{B}) \xrightarrow{\mathrm{HT}_B} H^0(X, \Omega^1 \otimes B) \to 0$$

3.5 Construction of the p-adic Simpson functor

According to Proposition 3.2, the datum of the lift $\mathbb{X}$ induces a splitting of HT_B

$$s_{\mathbb{X}} : H^0(X, \Omega^1 \otimes B) \to H^1_v(X, B).$$

It remains to explain the role of the exponential. This is done by the following result in the theory of rigid group varieties (whose proof is tied to p-adic Hodge theory of p-divisible groups):

Theorem 3.5 (Faltings [14], Fargues [16], H.–Werner–Zhang [26]). *Let G be a rigid group for which there is an open subgroup on which $[p]$ is surjective. Then the datum of an exponential induces a functorial Lie group exponential*

$$\mathrm{Exp}_G : \mathrm{Lie}\, G \to G(\mathbb{C}_p).$$

It is easy to deduce from Theorem 3.4 that the rigid group $G = \mathrm{vPic}_B$ satisfies the assumptions of Theorem 3.5. We conclude that there is a natural exponential map

$$\mathrm{Exp}_{\mathrm{vPic}_B} : H^1_v(X, \mathcal{B}^\times) \to H^1_v(X, \mathcal{B}).$$

In summary, we thus obtain a splitting of HTlog on $\mathbb{C}_p$-points:

$$\begin{tikzcd} H^1_v(X, \mathcal{B}^\times) \arrow[r, "\mathrm{HTlog}"] & H^0(X, \Omega^1 \otimes B) \arrow[d, equal] \\ H^1_v(X, \mathcal{B}) \arrow[u, "\mathrm{Exp}_{\mathrm{vPic}_B}"] \arrow[r] & H^0(X, \Omega^1 \otimes B) \arrow[l, bend left, "s_{\mathbb{X}}"] \end{tikzcd}$$

Now we use the canonical element $\tau \in H^0(X, \Omega^1 \otimes B)$ corresponding to the natural map $\mathrm{Sym}\Omega^\vee \to B$ and set

$$\mathcal{L} := \mathrm{Exp}(s_{\mathbb{X}}(\tau)) \in H^1_v(X, \mathcal{B}^\times).$$

This gives the desired invertible $\mathcal{B}$-module that we use to define the p-adic Simpson functor!

Caveat: This only defines an *isomorphism class* of $\mathcal{L}$. To obtain a canonical and functorial representative $\mathcal{L}$, we use in [25, §3] a "rigidification at every point of X", a method that we learnt from Faltings.

In summary, the p-adic Simpson functor from Higgs bundles to v-vector bundles is now given by sending a Higgs bundle (E, θ) with associated algebra B to

$$\nu^* E \otimes_{\mathcal{B}} \mathcal{L}.$$

In fact, one can show that the local correspondence Theorem 1.11 can be obtained by a similar twisting construction. In particular, this shows:

Corollary 3.6 (Proposition 4.7). *The p-adic Simpson functor is compatible with the local correspondences after localisation.*

3.6 The canonical Higgs field on v-vector bundles

At this point we have given a functor from Higgs bundles to v-vector bundles. In order to see that this is an equivalence, we now construct a functor in the other direction: from v-vector bundles to Higgs bundles. For this we need to explain how we can twist v-vector bundles by $\mathcal{B}$-modules. This relies on the canonical Higgs field of v-vector bundles, first described by Pan and Rodríguez-Camargo in the context of Geometric Sen theory.

Recall: The functor

$$\{\mathbb{C}_p\text{-local systems}\} \longrightarrow \{\text{v-vector bundles}\}, \quad \mathbb{L} \mapsto \mathbb{L} \otimes_{\mathbb{C}_p} \mathcal{O}$$

is a p-adic analogue of the Riemann–Hilbert functor

$$\{\mathbb{C}\text{-local systems}\} \longrightarrow \{\text{holomorphic vector bundles}\}, \quad \mathbb{L} \mapsto \mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}$$

Over $\mathbb{C}$, the vector bundle $V = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}$ comes equipped with a natural additional structure: the differential $d : \mathcal{O} \to \Omega^1$ induces a flat connection $\nabla : V \to V \otimes \Omega^1$.

Over $\mathbb{C}_p$, we cannot expect to obtain such a connection because $d : \mathcal{O} \to \Omega^1$ does not extend to a morphism of sheaves on X_v. Nevertheless, it turn out that there is an analogue of the flat connection from the Riemann–Hilbert correspondence in this context. In fact, it is conceptually more closely related to p-curvature in characteristic p:

Theorem 3.7 (Pan [31, §3], Rodríguez Camargo [32], [25, Theorem 4.8])**.** *Let S be any smooth rigid space over a perfectoid field over $\mathbb{Q}_p$. Then any v-vector bundle V on S is endowed with a* ***canonical Higgs field***

$$\theta_V : V \longrightarrow V \otimes_{\mathcal{O}_X} \nu^*\Omega^1$$

that is uniquely determined by the following properties:

1. *θ_V is functorial in S and V,*
2. *$\theta_V = 0 \iff V$ is analytic-locally trivial.*
3. *If V is small, then via the local p-adic Simpson correspondence of Theorem 1.11*

$$\left\{\begin{matrix}\textit{small}\\ \textit{v-vector bundles}\end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix}\textit{small}\\ \textit{Higgs bundles}\end{matrix}\right\}$$

it corresponds to the morphism of Higgs bundles tautologically defined by θ:

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[d, "\theta"] & E \otimes \Omega^1 \arrow[d, "\theta"] \\ E \otimes \Omega^1 \arrow[r, "\theta"] & E \otimes \Omega^1 \otimes \Omega^1 \end{tikzcd}$$

3.7 From v-vector bundles to Higgs bundles

Equipped with the canonical Higgs field, we can now obtain an analogue of abelianization for v-vector bundles: For any v-vector bundle V, the canonical Higgs field defines a canonical and functorial morphism

$$\theta_V : \mathrm{Sym}\Omega^\vee \to \nu_*\mathrm{End}(V).$$

It is easy to see locally that its image is a coherent $\mathcal{O}_X$-algebra B. The natural morphism $B \to \nu_*\mathrm{End}(V)$ induces by adjunction a morphism

$$\mathcal{B} := \nu^*B \to \mathrm{End}(V).$$

Via the construction of the previous section, lift and exponential induce an invertible $\mathcal{B} := \nu^* B$-module $\mathcal{L}$. We then define the p-adic Simpson functor to be

$$S(V) := \nu_*\Big((V, \theta_V) \otimes_{\mathcal{B}} \mathcal{L}^{-1}\Big)$$

The remaining calculation is now:

Proposition 3.8 ([25, Proposition 4.13.]). *$S'(V) := V \otimes_{\mathcal{B}} \mathcal{L}^{-1}$ is an analytic-locally trivial v-vector bundle on X.*

Proof. It is clear that $S'(V)$ is a v-vector bundle endowed with a Higgs field. It suffices to prove that $S'(V)$ is étale-locally trivial. For this we use Corollary 3.6, which says that the local correspondence Theorem 1.11 is also given by twists of $\mathcal{B}$-modules. This shows that étale-locally where V is small, it is associated to a Higgs bundle via twisting with $\mathcal{L}$. Clearly, $-\otimes_B \mathcal{L}^{-1}$ undoes this twisting, meaning that $S'(V)$ is étale-locally isomorphic to an étale Higgs bundle. Hence it is étale-locally trivial. □

This shows that $S(V) := \nu_* S'(V)$ is a Higgs bundle on X of the same rank as V.

This defines the p-adic Simpson functor on objects. To show that this construction is functorial, it suffices to note that there is some flexibility in the choice of B: Instead of defining it to be the image of $\operatorname{Sym}\Omega^\vee \to \operatorname{End}(E)$, we can for any second Higgs bundle (E', θ') with a morphism $(E, \theta) \to (E', \theta')$ replace B by the image of $\operatorname{Sym}\Omega^\vee \to \operatorname{End}(E) \oplus \operatorname{End}(E')$, which is still an $\mathcal{O}_X$-coherent commutative algebra. Moreover, the construction of $\mathcal{L}$ was natural in B. From this one easily verifies that the construction in both directions is functorial.

This finishes the construction of the p-adic Simpson functor!

3.8 Outlook

1. The construction of the p-adic Simpson functor was ultimately in terms of twisting by invertible modules. In upcoming work of Bhatt–Zhang [5], this is explained in a very nice geometric way by way of a canonical $\mathbb{G}_m$-gerbe on the cotangent bundle, called the Simpson gerbe. From this perspective, the choices in the p-adic Simpson correspondence are related to the fact that the Simpson gerbe is not split.

   In work-in-progress of Bhatt–Kanaev–Mathew–Vologodsky–Zhang, it will moreover be explained how this can be related to mod-p non-abelian Hodge theory, namely the gerbes of differential operators appearing in the work of Ogus-Vologodsky [30].

2. We saw in §2 that for line bundles, there is a moduli-theoretic incarnation of the p-adic Simpson correspondence: The moduli space of v-line bundles is an étale twist of the moduli space of Higgs bundles. This in particular explains why choices are necessary to formulate the p-adic Simpson correspondence: They are used to trivialise the twist on points.

   In joint work with Daxin Xu [27], we showed that a very similar picture holds more generally in higher rank: The stack of v-vector bundles of rank n is a twist of the stack of Higgs bundles of rank n, at least for curves. [Here we could insert a reference to Daxin's contribution to proceedings]

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