Chapter 1

Perfectoid Spaces

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The purpose of this course is to give an introduction to the theory of **perfectoid spaces**. We first give a summary of some basic constructions in perfectoid geometry. Second, we aim to give an idea of how perfectoid spaces arise "in nature" and how they are used: We discuss the pro-étale site and applications to p-adic Hodge theory, namely the construction of the Hodge–Tate spectral sequence. Our focus lies on discussing motivation, examples and the big picture, and less on completeness or technical details. But we at least give key ideas for all main proofs.

1.1 Lecture 1: Perfectoid fields

Once and for all, let us fix a prime number $p \in \mathbb{N}$.

Perfectoid spaces are a class of geometric objects in non-archimedean geometry over $\mathbb{Z}_p$ that were introduced by Peter Scholze in his 2012 PhD thesis at the University of Bonn [Sch12]. They have had a great impact on p-adic geometry over the course of the past decade. By way of motivation, we start in Lecture 1 by introducing the very first instance of "perfectoid objects", namely perfectoid fields. We will use these to motivate some basic "perfectoid" constructions, especially tilting and untilting.

1.1.1 Infinite ramification in valued fields

1.1.1.1 The field of Laurent series over $\mathbb{F}_p$. Let us start our journey with the ring of formal power series $R = \mathbb{F}_p[[t]]$ over the finite field $\mathbb{F}_p$. This is a discrete valuation ring with uniformizer t. We recall what this means concretely: R is a commutative ring with a "discrete valuation". namely the function

$$v_t : R \longrightarrow \mathbb{Z} \cup \{\infty\}, \quad f = \sum_{n=0}^{\infty} a_n t^n \mapsto \inf\{n \in \mathbb{Z}_{\geq 0} \text{ s.t. } a_n \neq 0\}$$

satisfies the axioms of a valuation from [Ber, §1.2]. That t is a uniformizer means that $v(t) = 1$. In fact, R is a *complete* discrete valuation ring, since we moreover have

$$R = \varprojlim_{n \in \mathbb{N}} R/t^n.$$

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As is true for any discrete valuation ring, the fact that t is not a unit implies that $K := R[t^{-1}]$ is a valued field: the **field of Laurent series** $K = \mathbb{F}_p((t))$. Its elements can be written as formal series

$$f = \sum_{n\geq m}^{\infty} a_n t^n \text{ for some } m \in \mathbb{Z},$$

with valuation still given by

$$v_t : K \to \mathbb{Z} \cup \{\infty\}, \quad f \mapsto \inf\{n \in \mathbb{Z} \text{ s.t. } a_n \neq 0\}.$$

Since R is t-adically complete, K is a non-archimedean field (as defined in [Ber, Example 1.1.2.3]) of characteristic p with finite residue field. Hence it is a **local field**. If you are geometrically minded, you can think of K as the functions on the punctured infinitesimal neighbourhood of

$$0 \in \mathbb{A}^1_{\mathbb{F}_p} := \mathrm{Spec}(\mathbb{F}_p[t]).$$

1.1.1.2 Reminders on Frobenius and ramification. Like any ring of characteristic p, the field $K := \mathbb{F}_p((t))$ has the field endomorphism

$$F : K \to K, \quad x \mapsto x^p,$$

called the Frobenius endomorphism, or simply **Frobenius**. Concretely, in terms of Laurent series, this sends

$$\sum a_n t^n \mapsto \sum a_n t^{np}$$

because $a^p = a$ for any $a \in \mathbb{F}_p$. Observe that as a morphism of rings, F is finite of degree p and generated by t, namely a basis is given by $1, t, \dots, t^{p-1}$. We can thus regard $F : K \to K$ as a finite field extension.

As a finite field extension, the Frobenius is **ramified**. What does this mean?

- Geometrically, F is the generic fibre of the completion of the morphism

  $$\mathbb{A}^1_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}, \quad t \mapsto t^p$$

  at the origin $0 \in \mathbb{A}^1_{\mathbb{F}_p}$. Note that the degree of the morphism is p, but 0 is the only geometric point over 0. Hence the morphism is ramified over 0. More precisely, $F : K \to K$ is a **totally ramified** extension of ramification index p.
- Algebraically, F being ramified means the following: It is convenient to renormalise the codomain of F via the rescaling isomorphism

  $$r : \mathbb{F}_p((t)) \xrightarrow{\sim} \mathbb{F}_p((t^{1/p})), \quad t \mapsto t^{\frac{1}{p}},$$

  where on the right, we consider $t^{1/p}$ as a formal variable that satisfies $(t^{1/p})^p = t$. Then we can identify F with the canonical inclusion

  $$\iota := r \circ F : \mathbb{F}_p((t)) \hookrightarrow \mathbb{F}_p((t^{1/p})).$$

To make this compatible with valuations, we renormalise the valuation on the codomain:

$$v_t : \mathbb{F}_p((t^{1/p})) \longrightarrow \tfrac{1}{p}\mathbb{Z} \cup \{\infty\}, \quad t^{1/p} \mapsto \tfrac{1}{p}.$$

The morphism F is then compatible with the valuations, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}_p((t)) & \overset{\iota}{\hookrightarrow} & \mathbb{F}_p((t^{1/p})) \\ \downarrow{\scriptstyle v_t} & & \downarrow{\scriptstyle v_t} \\ \mathbb{Z} \cup \{\infty\} & \hookrightarrow & \tfrac{1}{p}\mathbb{Z} \cup \{\infty\} \end{array} \tag{1.1}$$

It follows that the quotient of the value groups on the bottom line is given by $\mathbb{Z}/p\mathbb{Z}$ (recall that the value group is what's left when you forget about "$\cup\{\infty\}$"). The order $p = \#\mathbb{Z}/p\mathbb{Z}$ of this quotient is the **ramification index** $e(L|K)$ of ι, where $L = \mathbb{F}_p((t^{1/p}))$. In this language, the field extension ι is **totally ramified** because

$$e(L|K) = [L : K].$$

This is the algebraic (valuation theoretic) sense in which F is totally ramified.

1.1.1.3 The Frobenius tower. We now iterate the Frobenius morphism, that is, we form the tower

$$\mathbb{F}_p((t)) \xrightarrow{F} \mathbb{F}_p((t)) \xrightarrow{F} \mathbb{F}_p((t)) \to \dots$$

Iterating the rescaling procedure via ι from before, this tower becomes isomorphic to

$$\mathbb{F}_p((t)) \hookrightarrow \mathbb{F}_p((t^{1/p})) \hookrightarrow \mathbb{F}_p((t^{1/p^2})) \hookrightarrow$$

We now form the colimit of this diagram and arrive at the field

$$K_\infty := \varinjlim_{n\in\mathbb{N}} \mathbb{F}_p((t^{1/p^n})).$$

As the renormalised valuations from (1.1) are compatible in the tower, K_∞ inherits a natural t-adic valuation

$$v_t : K_\infty \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \cup \{\infty\}$$

defined as the colimit of the valuations of $\mathbb{F}_p((t^{1/p^n}))$ for each n. Explicitly, every element $f \in K_\infty$ can be written uniquely as a formal sum

$$f = \sum_{m\in\mathbb{Z}[\frac{1}{p}]} a_m t^m$$

such that $a_m = 0$ for m small enough. Then $v_t(f) = \inf\{m \text{ s.t. } a_n \neq 0\}$.

This makes K_∞ a valued field. However, this valued field is no longer complete!

Example 1.1.1. The sequence $(f_n := \sum_{m=0}^{n} t^{m+\frac{1}{p^m}})_{n\in\mathbb{N}}$ converges in K_∞, because the difference $f_n - f_k$ is divisible by t^n for any $k, n \in \mathbb{N}$ with $k \geq n$. In terms of formal power series, the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ should be $\sum_{m=0}^{\infty} t^{m+\frac{1}{p^m}}$, but this is not contained in any of the K_n, hence it cannot be contained in the colimit K_∞.

1.1.1.4 Our first perfectoid field.

Definition 1.1.2. Let $\mathbb{F}_p((t^{1/p^\infty}))$ be the completion of K_∞ with respect to the t-adic valuation. Explicitly, we can construct this by first forming the valuation ring of K_∞:

$$R_\infty := \varinjlim_n \mathbb{F}_p[[t^{1/p^n}]].$$

Then

$$\mathbb{F}_p[[t^{1/p^\infty}]] := \varprojlim_{d\in\mathbb{N}} R_\infty/t^d, \quad \mathbb{F}_p((t^{1/p^\infty})) := \mathbb{F}_p[[t^{1/p^\infty}]][\tfrac{1}{t}].$$

Remark 1.1.3. The notation $\mathbb{F}_p((t^{1/p^\infty}))$ is arguably a bit ambiguous, as one might be led to think that it refers to K_∞. However, this notation is a very common convention.[1]

Exercise 1.1.4.

(1) Give an explicit description of $\mathbb{F}_p[[t^{1/p^\infty}]]$ in terms of formal power series of the form $\sum_{m\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_m t^m$ with $a_m \in \mathbb{F}_p$. Which formal series indexed by $\mathbb{Z}[\frac{1}{p}]_{\geq 0}$ are included in $\mathbb{F}_p[[t^{1/p^\infty}]]$? Give an explicit example of a formal series of the form $\sum_{m\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_m t^m$ with $a_m \in \mathbb{F}_p$ that is not contained in $\mathbb{F}_p[[t^{1/p^\infty}]]$.

(2) Give an explicit description of $\mathbb{F}_p((t^{1/p^\infty}))$ in terms of formal power series of the form $\sum_{m\in\mathbb{Z}[\frac{1}{p}]} a_m t^m$ with $a_m \in \mathbb{F}_p$.

(3) Convince yourself that R_∞ is a valuation ring because it is a filtered colimit of valuation rings. Describe the maximal ideal $\mathfrak{m}_\infty$ of R_∞. Prove that the residue field of $\mathbb{F}_p((t^{1/p^\infty}))$ is still $\mathbb{F}_p$. So extending from $\mathbb{F}_p((t))$ to $\mathbb{F}_p((t^{1/p^\infty}))$ did not change the residue field.

(4) Note that we completed R_∞ with respect to the ideal (t) spanned by the pseudo-uniformizer t, not with respect to the maximal ideal $\mathfrak{m}_\infty$ of R_∞, as we would usually do when dealing with discrete valuation rings. What would happen if we took the maximal ideal instead, i.e. what is $\varprojlim_{d\in\mathbb{N}} R_\infty/\mathfrak{m}_\infty^d$?

(5) Show that $\mathbb{F}_p[[t^{1/p^\infty}]]$ is a complete valuation ring. Describe its maximal ideal. Show that the maximal ideal is not finitely generated. In particular, $\mathbb{F}_p[[t^{1/p^\infty}]]$ is not Noetherian!

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[1] Technical remark: Categorically speaking, this is because there is only one reasonable meaning to $\mathbb{F}_p((t^{1/p^\infty}))$ when we work in the category of non-archimedean fields. Indeed, recall that by definition in [Ber, Example 1.1.2.3], a non-archimedean field is complete. Hence, the categorical colimit in non-archimedean fields is the completed colimit $\mathbb{F}_p((t^{1/p^\infty}))$ from Definition 1.1.2.

As the maximal ideal of the valuation ring $\mathbb{F}_p[[t^{1/p^\infty}]]$ is no longer principal, it no longer makes sense to speak of uniformizers (i.e. a generator of $\mathfrak{m}_K$). Instead, we use:

Definition 1.1.5. When A is a valuation ring with valuation v, any $0 \neq \varpi \in A$ with $\varpi^n \to 0$ for $n \to \infty$ is called a **pseudo-uniformizer** of A. When the value group of A is contained in $\mathbb{R}$, this is equivalent to $0 < v(\varpi) < \infty$ for the valuation v of A.

This is a replacement for the concept of a uniformizer for cases when the maximal ideal is not finitely generated (which is going to happen a lot in these lectures).

Exercise 1.1.6. Check that for any pseudo-uniformizer ϖ, the fraction field of A is $A[1/\varpi]$, which is thus a valued field.

It is clear that t is a pseudo-uniformizer of $\mathbb{F}_p[[t^{1/p^\infty}]]$, hence $\mathbb{F}_p((t^{1/p^\infty}))$ is a complete valued field. Its valuation can still be explicitly described as

$$v : \mathbb{F}_p((t^{1/p^\infty})) \to \mathbb{Z}[\tfrac{1}{p}] \cup \{\infty\}, \qquad \sum_{m \in \mathbb{Z}[\frac{1}{p}]} a_m t^m \mapsto \min\{m \in \mathbb{Z}[\tfrac{1}{p}] \text{ s.t. } a_m \neq 0\}.$$

In particular, this is a non-archimedean field with valuation subring $\mathbb{F}_p[[t^{1/p^\infty}]]$.

Exercise 1.1.7. (1) Note that in the definition of v, we wrote min, not inf. Explain why this is well-defined, i.e. why the minimum is always attained. In particular, verify that the value group is really still $\mathbb{Z}[\frac{1}{p}]$, like for R_∞: This means that the value group did not get any larger when we took completions.

(2) Prove that a valuation ring whose value group contains $\mathbb{Z}[\frac{1}{p}]$ is non-Noetherian.

By definition of K_∞, it is identified with $\varinjlim_F \mathbb{F}_p((t))$. It follows that the Frobenius morphism of K_∞ is an isomorphism. Since completion of valued fields is functorial, it follows that

$$F : \mathbb{F}_p((t^{1/p^\infty})) \to \mathbb{F}_p((t^{1/p^\infty})), \quad t \mapsto t^p$$

is still an isomorphism. Here we use that the completion of the Frobenius on K_∞ is precisely the Frobenius on $\mathbb{F}_p((t^{1/p^\infty}))$, as the two coincide on the dense subfield K_∞. (If you have done Exercise 1.1.4.2, you can now also verify directly that F is an isomorphism.) We have thus constructed the first example of a **perfectoid field**!

Definition 1.1.8. A non-archimedean field L of characteristic p is called **perfectoid** if it is perfect, i.e. if its Frobenius morphism $F : L \to L$ is an isomorphism. Since any field homomorphism is injective, this is equivalent to asking that F is surjective.

This definition is preliminary, as we will later give a more general definition that extends to characteristic 0. We recall that a non-archimedean field is by definition a complete valued field whose valuation is non-trivial. In particular, fields like $\mathbb{F}_p$ are excluded: This field is *perfect*, but not *perfectoid*. Hence, in characteristic p, perfectoid is stronger than perfect: Additionally to the property that F is bijective, a perfectoid field always carries a non-trivial topology: They are objects of topological algebra, not just of abstract algebra. Why the name "perfectoid" makes sense will become clear when we consider characteristic 0.

1.1.1.5 Perfection of non-archimedean fields. Before we head to characteristic 0, let's take a step back. Question: How important was it really that the non-archimedean field we started with was $\mathbb{F}_p((t))$? Answer: Not at all! In fact, this kind of procedure works for any non-archimedean field:

Definition 1.1.9. Let K be any non-archimedean field of characteristic p. Then the **completed perfection** of K is defined as

$$K^{\mathrm{perf}} := (\varinjlim_F K)^\wedge.$$

Here, as usual, F denotes the Frobenius endomorphism, and the completion refers to the metric completion of $\varinjlim_F K$ for the metric topology induced by the valuation, suitably rescaled, exactly as for $\mathbb{F}_p((t))$. Explicitly, and more algebraically, for any pseudo-uniformizer $\varpi \in \mathcal{O}_K$, we can define

$$K^{\mathrm{perf}} := \Big(\varprojlim_d (\varinjlim_F \mathcal{O}_K)/\varpi^d\Big)[\tfrac{1}{\varpi}].$$

Exercise 1.1.10. (1) Show that K^{perf} is always a perfectoid field.

(2) Prove that $-^{\mathrm{perf}}$ defines a left adjoint to the forgetful functor

$$\{\text{perfectoid fields over } \mathbb{F}_p\} \to \{\text{non-archimedean fields over } \mathbb{F}_p\}.$$

(3) Show that the residue field of K^{perf} is the perfection of the residue field of K.

Generalising the construction of the perfectoid Laurent series ring $\mathbb{F}_p((t^{1/p^\infty}))$, iterating the Frobenius morphism thus defines an "infinitely ramified" tower over any non-archimedean field of characteristic p, and a perfectoid field sitting on top of it.

Of course the constructions in this section so far were specific to characteristic p since they crucially relied on the Frobenius morphism. Nevertheless, we will now discuss that there are similar "infinitely ramified" towers in characteristic 0.

1.1.1.6 A tower of infinite ramification in characteristic 0. Consider the field $K := \mathbb{Q}_p$ of p-adic numbers. This is a discretely valued field for the p-adic valuation $v = v_p$. For any $n \in \mathbb{N}$, we consider the local field

$$K_n := \mathbb{Q}_p(p^{1/p^n})|\mathbb{Q}_p.$$

This is a finite extension of $\mathbb{Q}_p$ of degree p^n, and again it is totally ramified in the sense of local fields. Namely, the unique extension of v_p to K_n is given by

$$\begin{array}{ccccc} \mathbb{Q}_p & \hookrightarrow & \mathbb{Q}_p(p^{1/p^n}) & & p^{1/p^n} \\ {\scriptstyle v_p}\downarrow & & {\scriptstyle v_p}\downarrow & & \big\downarrow \\ \mathbb{Z}\cup\{\infty\} & \hookrightarrow & \frac{1}{p^n}\mathbb{Z}\cup\{\infty\} & & \frac{1}{p^n} \end{array}$$

The ring of integers of $\mathbb{Q}_p(p^{1/p^n})$ is $\mathbb{Z}_p[p^{1/p^n}]$, the maximal ideal is (p^{1/p^n}).

As in characteristic p, let's now take the colimit over n:

Definition 1.1.11. Consider $\mathbb{Z}_p[p^{1/p^\infty}] := \varinjlim_{n\in\mathbb{N}} \mathbb{Z}_p[p^{1/p^n}]$ and let $\mathbb{Z}_p\langle p^{1/p^\infty}\rangle$ be its p-adic completion. This is a complete valuation ring, but no longer discrete or Noetherian (by Exercise 1.1.7). Note that p is a pseudo-uniformizer of this valuation ring. We let

$$\mathbb{Q}_p\langle p^{1/p^\infty}\rangle := \mathbb{Z}_p\langle p^{1/p^\infty}\rangle[\tfrac{1}{p}].$$

Since $\mathbb{Z}_p\langle p^{1/p^\infty}\rangle$ is a complete valuation ring, $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ is a complete non-archimedean field with valuation of the form

$$v : \mathbb{Q}_p\langle p^{1/p^\infty}\rangle \to \mathbb{Z}[\tfrac{1}{p}] \cup \{\infty\}., \quad p^{1/p^n} \mapsto 1/p^n$$

This will be our first example of a perfectoid field in characteristic 0.

Exercise 1.1.12. Prove that the residue field of the valuation ring $\mathbb{Z}_p\langle p^{1/p^\infty}\rangle$ is $\mathbb{F}_p$.

Even though there is no Frobenius morphism in this setting, the situation is quite similar to the one from §1.1.1.3 in characteristic p: We adjoined higher and higher p-power roots of the uniformizer and then took the completed colimit to get an infinitely ramified field extension. But this is so far just a vague analogy, there is at a first glance no immediate reason to expect that there is any more direct algebraic relation between $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $\mathbb{F}_p((t^{1/p^\infty}))$.

But a miracle happens: The relation between $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $\mathbb{F}_p((t^{1/p^\infty}))$ turns out to be much, much deeper than one can see from the surface! This is evidenced by the following result, which is more than 30 years older than the perfectoid theory:

Theorem 1.1.13 (Fontaine–Wintenberger, [FW79a]). *There is a natural*[2] *isomorphism between the absolute Galois groups*

$$\mathrm{Gal}(\overline{\mathbb{Q}_p\langle p^{1/p^\infty}\rangle}|\mathbb{Q}_p\langle p^{1/p^\infty}\rangle) = \mathrm{Gal}(\overline{\mathbb{F}_p((t^{1/p^\infty}))}|\mathbb{F}_p((t^{1/p^\infty}))).$$

Fontaine–Wintenberger's Theorem might be quite surprising when you first see it, as we usually think of arithmetics in general and Galois theory in particular as being quite different in characteristics 0 and p. So how does this equivalence work in practice? More precisely, since for any perfect field K, the Galois group of K governs the finite field extensions of K, we would like to understand how the finite Galois extensions of both fields are matched up.

The unramified extensions of any non-archimedean field correspond to the extensions of the residue field, which is $\mathbb{F}_p$ for both $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $\mathbb{F}_p((t^{1/p^\infty}))$, so these can be matched up in a natural way. Let us therefore focus on the ramified ones.

To give a first vague heuristic, the rough idea should be that we can "replace p by t in equations". For example, the extension of $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ defined by the polynomial $X^n - X - p$ might be sent to the finite field extension defined by the polynomial $X^n - X - t$. And the one defined by the minimal polynomial $X^2 - 5pX - p^{1/p^2}$ might be sent to the one defined by the minimal $X^2 - 5tX - t^{1/p^2}$, and so forth. But this is only a heuristic, there's no elementary way to see that this actually works! To understand why the heuristic does work out, it is helpful to take a further step back and generalise.

1.1.1.7 The cyclotomic tower. Are there any other fields besides $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ for which we have analogues of the Theorem of Fontaine–Wintenberger, identifying the Galois group of K with that of a perfectoid field in characteristic p? First-of-all, the choice of p as a uniformizer was arguably pretty arbitrary, and the same kind of construction works if we take $p + p^2$ instead, or $\log(1 + p)$, or in fact any other non-zero element of the maximal ideal of $\mathbb{Z}_p$. But in fact, there are other examples of totally ramified towers that do not arise from extracting p-power roots of an element in the maximal ideal. We now give a very interesting and useful example of this:

For simplicity, let's assume[3] that $p \neq 2$. Consider the cyclotomic extension

$$\mathbb{Q}_p(\zeta_{p^n})|\mathbb{Q}_p.$$

[2] Without further explanation, the word "natural" is nonsensical in this context: Indeed, the absolute Galois group is only well-defined up to non-canonical isomorphism unless we fix algebraic closures of both fields, which are a priori unrelated. However, it can be filled with mathematical content by saying that 1) there is a canonical and functorial way of identifying algebraic closures of the two fields and 2) this identification matches up the resulting Galois groups in a canonical way. We will see that this is indeed the case! More precisely, there is a canonical equivalence between the respective Galois categories of either field.

[3] This isn't actually seriously necessary, we only make this assumption to simplify the exposition, so we don't need to worry about some very minor complications

This is a totally ramified finite field extension of degree $\varphi(p^n) = p^{n-1}(p-1)$. In fact, it is Galois with group $(\mathbb{Z}/p^n\mathbb{Z})^\times$, namely the Galois action of any element $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ is given by

$$a * \zeta_{p^n} = \zeta_{p^n}^a .$$

The ring of integers of $\mathbb{Q}_p(\zeta_{p^n})$ is $\mathbb{Z}_p[\zeta_{p^n}]$, the residue field is again $\mathbb{F}_p$. The valuation is this time a bit harder to describe explicitly: It turns out that $\mathbb{Z}_p[\zeta_{p^n}]$ is a complete discrete valuation ring with uniformizer given by $1-\zeta_{p^n}$. One easily verifies that the Galois action is continuous with respect to the valuation topology.

Playing the same game as before, we can now first form the colimit

$$\mathbb{Q}_p(\zeta_{p^\infty}) := \varinjlim_{n\in\mathbb{N}} \mathbb{Q}_p(\zeta_{p^n})$$

and then complete with respect to the p-adic topology to get the field

$$\mathbb{Q}_p^{\mathrm{cyc}} := \mathbb{Q}_p(\zeta_{p^\infty})^\wedge = (\varprojlim_{d\in\mathbb{N}} \varinjlim_{n\in\mathbb{N}} \mathbb{Z}_p[\zeta_{p^n}]/p^d)[\tfrac{1}{p}].$$

This inherits a valuation

$$v : \mathbb{Q}_p^{\mathrm{cyc}} \to \mathbb{Z}[\tfrac{1}{p}] \cup \{\infty\}$$

making it a non-archimedean field. Exactly like $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$, the field $\mathbb{Q}_p^{\mathrm{cyc}}$ is an infinitely ramified non-archimedean extension of $\mathbb{Q}_p$. But in contrast to $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle|\mathbb{Q}_p$, it is even "infinite Galois" in the following sense: There is a natural Galois action on $\mathbb{Q}_p(\zeta_{p^\infty})$ with group $\mathbb{Z}_p^\times = \varprojlim_{n\in\mathbb{N}}(\mathbb{Z}/p^n\mathbb{Z})^\times$. By continuity, this extends to a continuous action of $\mathbb{Z}_p^\times$ on $\mathbb{Q}_p^{\mathrm{cyc}}$. In fact, there is a precise sense in which the morphism

$$\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cyc}}) \to \mathrm{Spa}(\mathbb{Q}_p)$$

is a Galois covering: It is a torsor under the profinite group $\mathbb{Z}_p^\times$ in a certain topology – we will make this precise later. This is a difference to the example of $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$.

Irrespective of this difference, the Fontaine–Wintenberger mystery repeats itself:

Theorem 1.1.14. *There is a "natural"*[4] *isomorphism of the absolute Galois groups*

$$\mathrm{Gal}(\overline{\mathbb{Q}_p^{\mathrm{cyc}}}|\mathbb{Q}_p^{\mathrm{cyc}}) = \mathrm{Gal}(\overline{\mathbb{F}_p((t^{1/p^\infty}))}|\mathbb{F}_p((t^{1/p^\infty}))).$$

In this case, it is already more difficult to try to write down a concrete correspondence of field extensions, since it is not immediately obvious that there is a pseudo-uniformizer $x \in \mathbb{Q}_p^{\mathrm{cyc}}$ that admits arbitrary p-th power roots, which can play the role of t in $\mathbb{F}_p((t^{1/p^\infty}))$. However, such an element turns out to exist:

[4]See footnote 1 for an explanation of what this means.

Exercise 1.1.15. (1) Prove that $x := \lim_{n\to\infty}(1-\zeta_{p^n})^{p^{n-1}(p-1)}$ is a well-defined element of $\mathbb{Q}_p^{\mathrm{cyc}}$, i.e. the sequence $(1-\zeta_{p^n})^{p^{n-1}(p-1)})_{n\in\mathbb{N}}$ converges.

(2) Show that x is a pseudo-uniformizer of $\mathbb{Q}_p^{\mathrm{cyc}}$ that has arbitrary p-power roots.

Remark 1.1.16. For the element x in Exercise 1.1.15 to exist, it is crucial that we complete p-adically after adjoining all ζ_{p^n}: The limit does not exist in $\mathbb{Q}_p(\zeta_{p^\infty})$.

In this sense, $\mathbb{Q}_p^{\mathrm{cyc}}$ does look a bit similar to a perfectoid field in characteristic p. At this point, we have now seen enough examples to understand the definition of:

1.1.2 Perfectoid fields

We are ready for the first key definition of this course!

Definition 1.1.17 ([Sch12]). A **perfectoid field** is a non-archimedean field K of residue field characteristic $p > 0$ with non-discrete value group such that the Frobenius

$$F : \mathcal{O}_K/p \to \mathcal{O}_K/p, \quad x \mapsto x^p$$

is surjective.

All three examples we discussed so far, $\mathbb{F}_p((t^{1/p^\infty}))$, $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $\mathbb{Q}_p^{\mathrm{cyc}}$ are perfectoid fields. For $K := \mathbb{Q}_p\langle p^{1/p^\infty}\rangle$, this follows from the following important exercise:

Exercise 1.1.18. (1) Verify that there is an isomorphism

$$\mathcal{O}_K/p = \mathbb{Z}_p[p^{1/p^\infty}]/p \xleftarrow{\sim} \mathbb{F}_p[t^{1/p^\infty}]/t, \quad p^{1/p^n} \leftarrow\!\shortmid t^{1/p^n}.$$

(2) Convince yourself that this shows that $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ is a perfectoid field.

Note that this makes slightly more precise the algebraic heuristic mentioned earlier that we “replace p^{1/p^n} by t^{1/p^n} in equations”.

Exercise 1.1.19. (1) Find an analogous isomorphism $\mathbb{Z}_p^{\mathrm{cyc}}/p \xleftarrow{\sim} \mathbb{F}_p[t^{1/p^\infty}]/t$ and use this to show that $\mathbb{Q}_p^{\mathrm{cyc}}$ is a perfectoid field. Hint: Use Exercise 1.1.15.

(2) Another important example of a perfectoid field: For any non-archimedean field K of residue characteristic p, the completion $\widehat{\overline{K}}$ of any algebraic closure $\overline{K}$ is complete and still algebraically closed (Krasner’s Lemma). Prove that $\widehat{\overline{K}}$ is perfectoid. In particular, this applies to $\mathbb{C}_p$, which is defined as the completion of an algebraic closure of $\mathbb{Q}_p$.

(3) A non-example: The field $\mathbb{Q}_p$ is not perfectoid. This is the kind of thing ruled out by the assumption that the value group is non-discrete. Prove that in fact, for any perfectoid field, the value group is p-divisible.

Remark 1.1.20. Note that in contrast to the definition of a perfect ring, the Frobenius homomorphism in Definition 1.1.17 is just assumed to be *surjective*, not *bijective*. In fact, for a perfectoid field K, the map $F : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is bijective if and only if K has characteristic p. The problem is that the kernel is too big: For example, for $K = \mathbb{Q}_p\langle p^{1/p^\infty}\rangle$, the element $p^{1/p}$ is non-zero but in the kernel of F.

That being said, we can always make F into an isomorphism by quotienting out by a larger ideal on the left hand side. For example,

$$\mathbb{Z}_p\langle p^{1/p^\infty}\rangle/p^{1/p} \to \mathbb{Z}_p\langle p^{1/p^\infty}\rangle/p, \quad x \mapsto x^p$$

is an isomorphism. See Exercise 1.1.33 below for more on this topic.

Remark 1.1.21. In scientific language, the suffix *-oid* (from Greek “eîdos”, meaning “form”) indicates that something “is not quite the same as but a bit similar to” or “has likeness to” whatever is described by the word before it. So a “humanoid” resembles a human, an “affinoid” resembles an affine scheme, and the definition of a perfectoid field resembles that of a perfect field. Saying “-oid” is basically the intellectual way of saying “-ish”. Maybe don’t call them “perfect-ish spaces”, though.

1.1.2.1 Inverse perfection, $\flat$ and $\sharp$. In Section 1.1.1.5, we have considered the perfection functor $A \mapsto \varinjlim_F A$ defined on the category of $\mathbb{F}_p$-algebras. We now turn the arrow around and consider the “inverse perfection”:

Exercise 1.1.22. (1) Let A be any $\mathbb{F}_p$-algebra. Show that the limit $\varprojlim_F A$ of the diagram

$$A \xleftarrow{[p]} A \xleftarrow{[p]} \dots$$

is a perfect $\mathbb{F}_p$-algebra. This is called the “inverse perfection”.

(2) Show that the perfection $A \mapsto \varinjlim_F A$ is a left adjoint for the forgetful functor

$$\{\text{perfect } \mathbb{F}_p\text{-algebra}\} \to \{\mathbb{F}_p\text{-algebra}\}$$

while the inverse perfection $A \mapsto \varprojlim_F A$ is a right adjoint of the forgetful functor. Deduce the respective universal properties of $\varinjlim_F A$ and $\varprojlim_F A$.

The key observation is now that we can in fact recover $\mathbb{F}_p[[t^{1/p^\infty}]]$ from

$$\mathbb{F}_p[[t^{1/p^\infty}]]/t \cong \mathbb{Z}_p^{\mathrm{cyc}}/p$$

by forming its inverse perfection: Namely, observe that the inverse perfection of $\mathbb{F}_p[[t^{1/p^\infty}]]/t$ is $\mathbb{F}_p[[t^{1/p^\infty}]]$, via the map

$$\varprojlim_F \mathbb{F}_p[[t^{1/p^\infty}]]/t \xrightarrow{(F^n)_{n\in\mathbb{N}}} \varprojlim_{n\in\mathbb{N}} \mathbb{F}_p[[t^{1/p^\infty}]]/t^{p^n} = \mathbb{F}_p[[t^{1/p^\infty}]].$$

All in all, we have thus found a concrete way to construct $\mathbb{F}_p((t^{1/p^\infty}))$ out of $\mathbb{Q}_p^{\mathrm{cyc}}$. Namely, we can (so far rather vaguely) sketch this "evolution" in the following steps:

$$\mathbb{Q}_p^{\mathrm{cyc}} \rightsquigarrow \mathbb{Z}_p^{\mathrm{cyc}} \rightsquigarrow \mathbb{Z}_p^{\mathrm{cyc}}/p \rightsquigarrow \mathbb{F}_p[[t^{1/p^\infty}]]/t \rightsquigarrow \mathbb{F}_p[[t^{1/p^\infty}]] \rightsquigarrow \mathbb{F}_p((t^{1/p^\infty})),$$

where the first step is forming the ring of integral elements, the third step is in fact an isomorphism, and the fourth arrow is the inverse perfection.

As we shall see next, this works very generally, yielding a functorial construction!

Proposition 1.1.23. *Let K be any perfectoid field. Consider*

$$\mathcal{O}_{K^\flat} := \varprojlim_F \mathcal{O}_K/p$$

endowed with the inverse limit topology, where $\mathcal{O}_K/p$ carries the discrete topology. Then as a topological ring, $\mathcal{O}_{K^\flat}$ is a complete valuation ring of characteristic p. The fraction field

$$K^\flat := \mathrm{Frac}(\mathcal{O}_{K^\flat})$$

is a perfectoid field of characteristic p.

To prove the Proposition, we first show the following lemma:

Lemma 1.1.24. *The natural map of sets*

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_K \to \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p = \mathcal{O}_{K^\flat}$$

is bijective and multiplicative. It's inverse is given for any $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathcal{O}_K/p$ by choosing any lifts $\tilde{x}_n$ of x_n to $\mathcal{O}_K$ and sending these to

$$(\lim_{m\in\mathbb{N}} \tilde{x}_{n+m}^{p^m})_{n\in\mathbb{N}}.$$

Proof. We have $\tilde{x}_{n+1}^p = \tilde{x}_n + pa$ for some $a \in \mathcal{O}_K$, and hence $\tilde{x}_{n+1}^{p^{n+1}} = (\tilde{x}_n + pa)^{p^n} \equiv \tilde{x}^{p^n} \bmod p^n$. It follows that $(\tilde{x}_n^{p^n})_{n\in\mathbb{N}}$ is a Cauchy sequence.

Exercise 1.1.25. (1) Verify the remaining statements of the Lemma.

(2) Show that for any $\varpi \in \mathcal{O}_K$ with $p \in \varpi\mathcal{O}_K$, the following map is also an isomorphism:

$$\varprojlim_F \mathcal{O}_K/p \to \varprojlim_F \mathcal{O}_K/\varpi. \qquad \blacksquare$$

Proof of Proposition 1.1.23. : It is clear that $F : \mathcal{O}_{K^\flat} \to \mathcal{O}_{K^\flat}$ is bijective, so $\mathcal{O}_{K^\flat}$ is a perfect ring of characteristic p. We endow it with the inverse limit topology.

Let $\varpi \in \mathcal{O}_K$ such that $0 < v(\varpi) < 1$, this exists by the assumption that the value group is not discrete. Since $F : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective, for every $n \in \mathbb{N}$ we can find $\varpi_n \in \mathcal{O}_K$ such that $\varpi_n^p \equiv \varpi_{n-1} \mod p$ for all n. Thus $\varpi^\flat := (\varpi_n)_{n\in\mathbb{N}}$ defines an element of $\mathcal{O}_{K^\flat}$.

One verifies directly that $\mathcal{O}_{K^\flat}/\varpi^\flat = \mathcal{O}_K/\varpi$. Since $\mathcal{O}_{K^\flat}$ is perfect, it follows that

$$\mathcal{O}_{K^\flat} = \varprojlim_F \mathcal{O}_{K^\flat}/\varpi^\flat = \varprojlim_n \mathcal{O}_{K^\flat}/\varpi^{\flat p^n},$$

hence $\mathcal{O}_{K^\flat}$ is ϖ-adically complete.

Finally, it follows from the multiplicative bijection $\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ that $\mathcal{O}_{K^\flat}$ is integral and the map

$$\mathcal{O}_{K^\flat}[\tfrac{1}{\varpi^\flat}] = \varprojlim_{x\mapsto x^p} K$$

is also a multiplicative bijection. Since on the right hand side, every non-zero element has an inverse, this shows that $K^\flat = \mathcal{O}_{K^\flat}[\frac{1}{\varpi^\flat}]$ is a field. Finally, for any $0 \neq y \in K^\flat$, we have $y \in \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ or $y^{-1} \in \varprojlim_{x\mapsto x^p} \mathcal{O}_K$. Hence $\mathcal{O}_{K^\flat}$ is a complete valuation ring with pseudo-uniformizer $\varpi^\flat$. ∎

It is clear from the definition that $-^\flat$ is functorial. We have thus defined:

Definition 1.1.26. The **"tilting functor"** is the functor

$$\begin{array}{rcl} \{\text{perfectoid fields in characteristic } 0\} & \to & \{\text{perfectoid fields in characteristic p}\} \\ K & \mapsto & K^\flat := \mathrm{Frac}(\mathcal{O}_{K^\flat}) \end{array}$$

Exercise 1.1.27. (1) If K is a perfectoid field of characteristic p, show $K^\flat = K$.

(2) Use Exercise 1.1.18 to prove that $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle^\flat = \mathbb{F}_p((t^{1/p^\infty}))$.

(3) Use Exercise 1.1.19 to prove that $(\mathbb{Q}_p^{\mathrm{cyc}})^\flat = \mathbb{F}_p((t^{1/p^\infty}))$. In particular, it follows from this and part (2) of this Exercise that $\flat$ is not fully faithful! (Indeed, as a bonus exercise, prove that $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ is not isomorphic to $\mathbb{Q}_p^{\mathrm{cyc}}$.)

Example 1.1.28. We will later see in Theorem 1.1.39 that the tilt of any algebraically closed perfectoid field is still algebraically closed. For example, $\mathbb{C}_p^\flat$ is the completion of an algebraic closure of $\mathbb{F}_p((t^{1/p^\infty}))$. This will clarify the naturality in Theorem 1.1.13 (see Corollary 1.1.40 for details on this).

In order to give another important example of perfectoid fields, let us already mention the following result, which we will discuss in greater generality in section 1.2 (see Corollary 1.2.43)

Theorem 1.1.29. *Let K be a perfectoid field and let $L|K$ be any finite field extension. Then L is again a perfectoid field.*

Having seen some examples, we now go on to study basic properties of perfectoid fields. The proof of Proposition 1.1.23 has also shown:

Corollary 1.1.30. *Let K be any perfectoid field. Then there is an isomorphism of topological groups*

$$K^{\flat\times} = \varprojlim_{x\mapsto x^p} K^{\times}.$$

The projection

$$\sharp : K^{\flat\times} = \varprojlim_{x\mapsto x^p} K^{\times} \to K^{\times}, \quad x = (x_n)_{n\in\mathbb{N}} \mapsto x^{\sharp} := x_1$$

is continuous with respect to the valuation topologies.

Corollary 1.1.31. *Let $\varpi^\flat \in \mathcal{O}_{K^\flat}$ be any pseudo-uniformizer of $K^\flat$. Then $\varpi := \varpi^{\flat\sharp}$ is a pseudo-uniformizer of K that admits arbitrary p-power roots in K.*

Proof. The element $\varpi^\flat$ admits a p^n-power root $\varpi^{\flat 1/p^n}$ since $K^\flat$ is perfectoid. Since $-^{\sharp}$ is a group homomorphism, the same is true for ϖ. Since $\varpi^{\flat n} \to 0$, it follows by continuity that $\varpi^n \to 0$, so ϖ is a pseudo-uniformizer of K. ∎

The existence of such a pseudo-uniformizer with arbitrary p-power roots is one of the many ways in which K behaves "similarly to a perfect field". We can now generalise the isomorphism $Z_p[p^{1/p^\infty}]/p = \mathbb{F}_p[t^{1/p^\infty}]/t$ from Exercise 1.1.18:

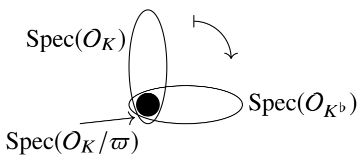
Exercise 1.1.32. Let $\varpi^\flat$ and ϖ be as in Corollary 1.1.31 and assume that $\varpi | p$, i.e. $p \in \varpi\mathcal{O}_K$. Then $\sharp$ induces an isomorphism of $\mathbb{F}_p$-algebras

$$\mathcal{O}_{K^\flat}/\varpi^\flat = \mathcal{O}_K/\varpi, \quad x \mapsto x^\sharp$$

We note that one can always arrange that $p \in \varpi\mathcal{O}_K$ by replacing ϖ with ϖ^{1/p^n} for n large enough. In fact, in this way, we may even assume that $\varpi^p|p$. Let us fix such a pair of pseudo-uniformizers $\varpi^\flat$ and $\varpi = \varpi^{\flat\sharp}$ with $\varpi^p|p$ in the following.

Exercise 1.1.33. A non-archimedean field K is a perfectoid field if and only if there is $\varpi \in K$ with $\varpi^p|p$ such that the map $\mathcal{O}_K/\varpi \xrightarrow{\sim} \mathcal{O}_K/\varpi^p$, $x \mapsto x^p$ is an isomorphism.

Remark 1.1.34. The name "tilt" comes from the following picture: Let ϖ, $\varpi^\flat$ be pseudo-uniformizers as above. We picture $\mathrm{Spec}(\mathcal{O}_K/\varpi)$ as being a "nilpotent thickening" of the closed point inside of $\mathrm{Spec}(\mathcal{O}_K)$, because it is sent to the closed point via the natural map and $\mathcal{O}_K/\varpi$ contains many nilpotent elements. The same applies to $\mathrm{Spec}(\mathcal{O}_{K^\flat}/\varpi^\flat) \to \mathrm{Spec}(\mathcal{O}_{K^\flat})$. But since $\mathcal{O}_{K^\flat}/\varpi^\flat = \mathcal{O}_K/\varpi$, we can imagine these thick points to be identified, as in the following picture (inspired by [SW20, p52]):



With this visualisation in mind, the idea is now that $-^\flat$ "tilts" a field from characteristic 0 into characteristic p. In line with this picture, the symbols are inspired by musical notation: The symbol $\flat$ (pronounced *flat*) lowers a note by a half-tone. The symbol $\sharp$ (pronounced *sharp*) raises a note by a half-tone. So $\flat$ "lowers K to characteristic p" and $\sharp$ "raises an element to characteristic 0" (maybe this will make a bit more sense when we apply $\sharp$ to algebras later on).

1.1.2.2 Historical remarks. Infinitely ramified fields like $\mathbb{F}_p((t^{1/p^\infty}))$ and $\mathbb{Q}_p^{\mathrm{cyc}}$ and $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ have been studied in number theory and arithmetic geometry for a long time, certainly > 50 years. But the definition of perfectoid fields due to Scholze gives a great framework to explain the behaviour of these fields that had been observed in the literature up to this point, and lends itself to generalisations that ultimately lead to the much more general concept of a perfectoid space. Let us give some examples for appearances of perfectoid fields in the literature that predate this definition:

- The field $\mathbb{Q}_p^{\mathrm{cyc}}$ makes an important appearance in one of the first works in the field: Tate's seminal article [Tat67]. This already contains some key observations e.g. about its Galois cohomology that we might today describe as being typically "perfectoid". We will discuss this link in some more detail in Section 1.4.4.2.
- More generally, perfectoid fields like $\mathbb{Q}_p^{\mathrm{cyc}}$ are important in p-adic Hodge theory, which is concerned with comparisons of cohomology theories for proper smooth p-adic varieties over $\mathbb{C}_p$, in the style of Hodge theory over $\mathbb{C}$. Examples include Sen theory and various works of Faltings. We will discuss this in more detail in the last lecture, when we discuss the Hodge–Tate spectral sequence.
- The field $\mathbb{Q}_p^{\mathrm{cyc}}$ also plays an important role in Iwasawa-theory.
- As mentioned before, the theory of perfectoid fields has its roots in the work of Fontaine–Wintenberger [FW79a][FW79b]. Going far beyond the isomorphism Theorem 1.1.13 mentioned above, they more generally define a notion of "arithmetically profinite fields" (APF), which are certain infinitely ramified Galois extensions of local fields, for example $\mathbb{Q}_p(\zeta_{p^\infty})|\mathbb{Q}_p$. To every such APF field L, they associate a valued field of characteristic p called the "field of norms". They show that this identifies the Galois groups of L and its field of norms, respectively. One can show that the completion $\widehat{L}$ of any APF field L is a perfectoid field, and the completion of the field of norms is the tilt $\widehat{L}^\flat$. Fontaine–Wintenberger discuss completions of APF fields in detail, including the multiplicative structure of $L^\flat$ (Corollary 1.1.30). In this context, they consider another construction that is crucial to the perfectoid theory:

1.1.2.3 Fontaine's θ-map. To any perfect ring R in characteristic p, we can functorially associate the ring of **Witt vectors** $W(R)$ (for an introduction to Witt vectors, see e.g. [Rab14]). We recall some basic properties of $W(R)$: It is a p-adically complete p-torsionfree $\mathbb{Z}_p$-algebra. For example, we have $W(\mathbb{F}_p) = \mathbb{Z}_p$. There is a natural isomorphism $W(R)/p = R$ as well as a canonical multiplicative section

$$[-] : R \to W(R)$$

(which is not additive: R is of characteristic p while $W(R)$ is p-torsionfree). With respect to this lift, any element in $x \in W(R)$ admits a series representation of the form

$$x = \sum_{n=0}^{\infty} [a_n] p^n$$

for some uniquely determined $a_n \in R$. We can use this to canonically identify

$$W(R) \cong \prod_{n=0}^{\infty} R$$

as *sets*. But writing down the addition and multiplication operations of $W(R)$ in terms of the right hand side is a bit involved and requires the so-called "ghost components". In fact, this concrete description is how $W(R)$ is usually defined.

We are going to apply this construction to perfectoid fields, as follows: Let K be any perfectoid field, then the tilt $K^\flat$ is perfectoid, hence perfect. Since $\mathcal{O}_{K^\flat} \subseteq K^\flat$ is a valuation ring, therefore integrally closed, it follows that $\mathcal{O}_{K^\flat}$ is a perfect $\mathbb{F}_p$-algebra. We can thus consider

$$W(\mathcal{O}_{K^\flat}).$$

This ring may first seem a bit scary, being large and non-Noetherian. But it's actually not that bad – one gets used to it! Its most important feature is that it allows us to "lift

$$\sharp : \mathcal{O}_{K^\flat} \to \mathcal{O}_K/p$$

to a ring homomorphism in characteristic 0": Namely, apart from the reduction map $W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_{K^\flat}$ and the canonical lift $[-] : \mathcal{O}_{K^\flat} \to W(\mathcal{O}_{K^\flat})$, we have the following very important homomorphism relating $W(\mathcal{O}_{K^\flat})$ to $\mathcal{O}_K$:

Proposition 1.1.35. *For any perfectoid field K, the map*

$$\theta : W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K, \quad \sum_{n=0}^{\infty} [a_n] p^n \mapsto \sum_{n=0}^{\infty} a_n^\sharp p^n,$$

*is a homomorphism of $\mathbb{Z}_p$-algebras, called **Fontaine's θ-map**. It is is surjective and the kernel of θ is a principal ideal.*

We do not discuss the proof of Proposition 1.1.35 here (you can find a proof in [Bha17, Proposition 6.1.8] or [SW20, Lemma 6.2.8]). We just note that the existence of θ does not follow immediately from the usual mapping property of $W(-)$ into p-strict rings (e.g. [Rab14, Theorem 6.6]) since $\mathcal{O}_K/p$ is not perfect (the Frobenius is only surjective, not necessarily bijective). One really has to do some non-trivial work to see that θ is a ring homomorphism.

Remark 1.1.36. While θ is most interesting when $\mathrm{Char}(K) = 0$, the statement is still correct when $\mathrm{Char}(K) = p$: Then $\sharp$ is the identity, so θ is just the canonical projection $W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_{K^\flat}$ to the first component. The kernel of θ is then generated by p.

Remark 1.1.37. Warning: One might be led to think that the same formula defines a map $W(K^\flat) \to K^\flat$, but this is not true! The problem is that the series on the right does not necessary converge in this case: There is nothing that stops the a_n from diverging faster in $K^\flat$ than p^n converges. A good analogy is the observation that $\mathcal{O}_K[[t]] \to \mathcal{O}_K$, $t \mapsto p$ is well-defined, but trying to extend this to $K[[t]] \to K$ does not work.

1.1.2.4 Our first version of the Tilting Equivalence. We have already seen that the tilting construction gives us a way to "go from characteristic 0 to characteristic p". Fontaine's map now gives us a way to "go into the other direction":

Definition 1.1.38. Let K be a perfectoid field and let $\varpi \in \mathcal{O}_K$ be a pseudo-uniformizer. Let L be any perfectoid field extension of $K^\flat$. The structure map $\mathcal{O}_{K^\flat} \to \mathcal{O}_L$ induces by functoriality of $W(-)$ a ring homomorphism $W(\mathcal{O}_{K^\flat}) \to W(\mathcal{O}_L)$. On the other hand, we can regard $\mathcal{O}_K$ as a $W(\mathcal{O}_{K^\flat})$-algebra via $\theta : W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$. We can therefore form the tensor product

$$\mathcal{O}_L^\sharp := W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat}),\theta} \mathcal{O}_K.$$

The **untilt** of L over K is

$$L^\sharp := \mathcal{O}_L^\sharp[\tfrac{1}{\varpi}].$$

We can use this to state the main theorem of the first lecture:

Theorem 1.1.39. *Let K be a perfectoid field. Then $-^\flat$ defines an equivalence*

$$\{\textit{perfectoid field extensions of } K\} \xrightarrow{\sim} \{\textit{perfectoid field extensions of } K^\flat\}.$$

The inverse functor is given by sending a perfectoid extension L of $K^\flat$ to $L^\sharp$. Moreover, the equivalence identifies the finite field extensions on both sides.

For the last sentence, we recall from Theorem 1.1.29 that any finite field extension is automatically perfectoid, so tilting it makes sense.

For the proof of the first part of Theorem 1.1.39, one checks directly that the two functors $-^\flat$ and $-^\sharp$ are quasi-inverses to each other. We defer the proof to later as we will repeat the argument in more detail in Theorem 1.2.25 below. The proof that $-^\flat$ and $-^\sharp$ identify finite extensions is more difficult. We will discuss this in section 1.2 where we also sketch a proof of Theorem 1.1.29. Once we have both of these results, note that as a Corollary, we obtain Theorem 1.1.13. Namely, we more generally have:

Corollary 1.1.40. *Let K be a perfectoid field. Then any algebraic closure $\overline{K}$ induces an algebraic closure $\overline{K}^\flat$ of $\overline{K}$, and a natural isomorphism* $\mathrm{Gal}(\overline{K}|K) = \mathrm{Gal}(\overline{K}^\flat|K^\flat)$.

Proof. Let $\overline{K}|K$ be the algebraic closure and let C be the completion of $\overline{K}$. Recall from Exercise 1.1.19.(2) that C is perfectoid. It follows by applying Theorem 1.1.39 to C that $C^\flat$ is algebraically closed. By functoriality, we obtain a map $K^\flat \to C^\flat$, hence we can define $\overline{K}^\flat$ to be the algebraic closure of $K^\flat$ inside $C^\flat$.

The proof is now essentially that $-^\flat$ and $-^\sharp$ identify the fibre functors of the respective Galois categories of K and $K^\flat$. Let us make this more explicit in down-to-earth terms: As any automorphism σ of $\overline{K}$ in $\mathrm{Gal}(\overline{K}|L)$ is automatically continuous, it extends uniquely to an automorphism $\widehat{\sigma} : C \to C$. Again by functoriality, this tilts to an automorphism

$$\widehat{\sigma}^\flat : C^\flat \to C^\flat$$

of $C^\flat$ that fixes $K^\flat$. Restricting to $\overline{K}^\flat$, we obtain the desired element $\sigma^\flat$ of $\mathrm{Gal}(\overline{K}^\flat|K^\flat)$.

Slightly rephrasing this construction, we can write any automorphism of $\overline{K}$ as an inverse system of automorphisms of its finite Galois subextensions. By applying the tilting equivalence, we obtain a compatible system of automorphisms of finite extensions of $K^\flat$ inside $C^\flat$. This defines a map

$$-^\flat : \mathrm{Gal}(\overline{K}|K) \to \mathrm{Gal}(\overline{K^\flat}|K).$$

From this construction, it is immediate that $-^\sharp$ defines an inverse mapping. ∎

Remark 1.1.41. Note that in the above discussion starting from Definition 1.1.38, we do not require K to be of characteristic 0. But if K has characteristic p, then the effect of tilting is trivial (see Exercise 1.1.27.(1)), and the same is true for untilting:

Exercise 1.1.42. Assume K has characteristic p. Verify that for any perfectoid field extension $L|K$, we have $L^\sharp = L$.

This verifies Theorem 1.1.39 in the easier case that K has characteristic p. Of course, the case that K has characteristic 0 is much more interesting. A related remark:

Remark 1.1.43. Warning: It is *not* true that $\sharp$ defines a functor from all perfectoid fields of characteristic p to perfectoid fields of characteristic 0. In fact, these categories are not equivalent via $\flat$, as we have $\mathbb{Q}_p^{\mathrm{cyc}\flat} = \mathbb{F}_p((t^{1/p^\infty})) = \mathbb{Q}_p\langle p^{1/p^\infty}\rangle^\flat$. The point is that the $\sharp$-functor only works after fixing some base field K' and fixing K such that $K' = K^\flat$. More precisely, we need the datum of the map θ as an input to define $\sharp$.

In short, tilting is defined for any perfectoid field, but untilting requires a base field which the untilt will live over (typically, this base field has characteristic 0). It is important to keep this difference in mind.

1.2 Lecture 2: Perfectoid K-algebras

1.2.1 Definition of perfectoid algebras

In §1.1, we discussed perfectoid fields (Definition 1.1.17) and the "tilting" functor

$$\{\text{perfectoid fields over } \mathbb{Q}_p\} \to \{\text{perfectoid fields over } \mathbb{F}_p\}, \quad K \mapsto K^\flat$$

(Definition 1.1.26). For a fixed perfectoid field K, we also saw the "untilting" functor

$$\{\text{perfectoid fields over } K^\flat\} \xrightarrow{\sim} \{\text{perfectoid fields over } K\}, \quad L \mapsto L^\sharp$$

(Definition 1.1.38). Our first goal in this second lecture is to extend these constructions from non-archimedean fields to a much larger class of topological rings.

How can we generalize the definition of perfectoid fields from fields to rings? We should replace non-archimedean fields by a category of topological rings. For this, we will use uniform complete Huber rings over K in the sense of [Ber, Definition 1.1.1.1] (we recall that a Huber ring R is uniform if R° is bounded, see [Hüb, §1.3] or [SW20, Definition 2.2.9]). The reasons are as follows:

(1) For perfectoid fields K, the ring of integers $\mathcal{O}_K$ played an important role in all constructions. Working with complete Huber rings R allows us to replace $\mathcal{O}_K$ by the ring of power-bounded elements R° of the K-Banach algebra R.

(2) The uniformity condition guarantees that R° is an adic ring, like $\mathcal{O}_K$.

(3) In the definition of perfectoid fields, we required the value group to be non-discrete. There are several ways to give analogues of this for Huber rings. The one that we shall use in these lectures is that R contains a perfectoid field K.

With this in mind, we get a straightforward generalization of Definition 1.1.17:

Definition 1.2.1. Fix a perfectoid field K. Then a **perfectoid K-algebra** is a uniform complete Huber ring R over K for which the following map is surjective:

$$F : R^\circ/p \to R^\circ/p, \quad x \mapsto x^p.$$

Recall from Corollary 1.1.31 that one can always find a pseudo-uniformizer $\varpi \in K$ that admits arbitrary p-th roots and such that $\varpi|p$. Let us fix such a ϖ. Then we have:

Exercise 1.2.2 ([Sch18, Remark 3.2]). Let R be a complete Huber ring over K, then due to the assumption that $\varpi|p$, the following map is a ring homomorphism:

$$\Phi : R^\circ/\varpi^{1/p} \to R^\circ/\varpi, \quad x \mapsto x^p.$$

Note that it is a homomorphism of $\mathcal{O}_K/\varpi$-algebras only after redefining the $\mathcal{O}_K$-algebra structure on the target to be given via the Frobenius $F : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi$.

Show that R is perfectoid if and only if R is uniform and Φ is bijective.

Remark 1.2.3. If you look closely at the original definition of perfectoid algebras in [Sch12], it might seem like there is a small difference to Definition 1.2.1: In [Sch12], the words "complete Huber ring over K" are replaced by "Banach K-algebra". But in fact there is no difference as these two notions are equivalent[5], see [SW20, p.11] or [Bha17, §5.2].

1.2.1.1 Examples of perfectoid algebras.

Example 1.2.4. As a first example, any perfectoid field extension L of K is a perfectoid K-algebra. Indeed, note that L is uniform because $L^\circ = \mathcal{O}_L$ is bounded.

Remark 1.2.5. Conversely, any perfectoid K-algebra which is a field is a perfectoid field [Ked18]. But this is hard to prove! The issue is that the norm of non-archimedean fields is required to be multiplicative, but the norm of K-Banach algebras is not.

Exercise 1.2.6. Prove that in characteristic p, a K-Banach algebra R is perfectoid if and only if it is perfect. The main work lies in proving that R is uniform.

Example 1.2.7. Consider the Tate algebra $K\langle X\rangle$. This is not perfectoid: The Frobenius $F : \mathcal{O}_K\langle X\rangle/p \to \mathcal{O}_K\langle X\rangle/p$ does not contain X in its image. But similarly to the ramified towers of fields from Lecture 1, there is a way to make $K\langle X\rangle$ perfectoid by considering lifts of Frobenius: Indeed, consider the direct system of K-algebras

$$K\langle X\rangle \to K\langle X^{1/p}\rangle \to \dots \to K\langle X^{1/p^n}\rangle \to \dots .$$

In terms of the associated adic spaces, this corresponds to an inverse system of discs

$$\dots \to \mathbb{B} \xrightarrow{x\mapsto x^p} \mathbb{B} \xrightarrow{x\mapsto x^p} \mathbb{B}.$$

We now take the completed colimit of the direct system of algebras, like we did for non-archimedean fields: Set

$$\mathcal{O}_K\langle X^{1/p^\infty}\rangle := (\varinjlim_{n\in\mathbb{N}} \mathcal{O}_K\langle X^{1/p^n}\rangle)^\wedge, \quad K\langle X^{1/p^\infty}\rangle := \mathcal{O}_K\langle X^{1/p^\infty}\rangle[\tfrac{1}{\varpi}]$$

where ϖ is any pseudo-uniformizer of K (if $\mathrm{Char}(K) = 0$, we can take $\varpi = p$).

Proposition 1.2.8. *$K\langle X^{1/p^\infty}\rangle$ is a perfectoid K-algebra.*

Proof. It is clear from the definition that $R := K\langle X^{1/p^\infty}\rangle$ is a complete Huber ring with ring of definition $R_0 := \mathcal{O}_K\langle X^{1/p^\infty}\rangle$. What is less obvious is that R is uniform.

We will see in Proposition 1.2.14 below a quick way to see this, but for now, to get a feeling for this statement, it is maybe more instructive to prove this by hand:

[5] And maybe the reason that K-Banach spaces are used in [Sch12] is that historically, this is the more classical notion. People weren't at all used to working with Huber pairs when this article was written.

We claim that in fact, we have $R^\circ = R_0$. This will show that R is uniform, and moreover that the map $F : R^\circ/p \to R^\circ/p$ is given by

$$\mathcal{O}_K/p[X^{1/p^\infty}] \to \mathcal{O}_K/p[X^{1/p^\infty}], \quad \sum a_n X^{1/n} \mapsto \sum a_n^p X^{p/n}. \tag{1.2}$$

Since K is perfectoid, $F : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective, thus (1.2) is surjective.

It remains to prove that $R^\circ = R_0$. Let us give an explicit way to see this, by arguing like in the proof that $K\langle X\rangle^\circ = \mathcal{O}_K\langle X\rangle$: It suffices to prove that the norm function

$$\| - \| : K\langle X^{1/p^\infty}\rangle \to \mathbb{R}_{\geq 0}, \quad \textstyle\sum_n a_n X^n \mapsto \max_n(|a_n|)$$

is in fact multiplicative: Indeed, let $f, g \in R$, then we clearly have $\|f \cdot g\| \leq \|f\| \cdot \|g\|$. To see that we have equality, we may rescale both $\|f\|$ and $\|g\|$ by elements in K to assume that $\|f\| = 1 = \|g\|$. It is clear from the definition of $\| - \|$ that this means that $f, g \in \mathcal{O}_K\langle X^{1/p^\infty}\rangle$. Let k be the residue field of K and consider the reduction map

$$\mathcal{O}_K\langle X^{1/p^\infty}\rangle \to k[X^{1/p^\infty}].$$

Let $\overline{f}$ and $\overline{g}$ be the images of f and g, then $\|f\| = 1 = \|g\|$ means that $\overline{f} \neq 0$ and $\overline{g} \neq 0$. Assume towards a contradiction that $\|fg\| < 1$, then all coefficients of fg lie in $\mathfrak{m}_K$, hence $\overline{fg} = 0$. But $k[X^{1/p^\infty}]$ is an integral domain, so this is a contradiction. ∎

Remark 1.2.9. In fact, if $(A_i)_{i\in I}$ is any filtered direct system of uniform Tate rings, the completion of its colimit $(A, A^+) = (\varinjlim_{i\in I}(A_i, A_i^\circ))^\wedge$ (in the sense of [Ber, §1.3.4]) is again uniform. Moreover, we have $A^\circ = A^+ = (\varinjlim_{i\in I} A_i^\circ)^\wedge$ if all the transition maps $A_i^\circ/\varpi \to A_j^\circ/\varpi$ are injective for $j \geq i$ in I, see [Heu19, Lemma A.2.2].

Example 1.2.10. As a slight variation, consider instead the algebra

$$K\langle X^{\pm 1}\rangle = K\langle X, Y\rangle/(XY - 1)$$

of the rigid unit annulus $\mathbb{T} = \mathrm{Spa}(K\langle X^{\pm 1}\rangle)$. Exactly as in the previous example, set

$$\mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle := (\varinjlim_{n\in\mathbb{N}} \mathcal{O}_K\langle X^{\pm 1/p^n}\rangle)^\wedge, \quad K\langle X^{\pm 1/p^\infty}\rangle := \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle[\tfrac{1}{\varpi}].$$

Exercise 1.2.11. Show that $K\langle X^{\pm 1/p^\infty}\rangle$ is a perfectoid K-algebra.

If $\mathrm{char}(K) = 0$, then $\mathbb{T} \xrightarrow{x\mapsto x^p} \mathbb{T}$ is finite étale, hence unramified. But the map $\mathcal{O}_K\langle X^{\pm 1}\rangle \to \mathcal{O}_K\langle X^{\pm 1}\rangle$, $X \mapsto X^p$ reduces to the relative Frobenius mod p and is thus totally ramified in the special fibre. This is exactly like in the §1.1.1.6, where $\mathbb{Q}_p(p^{1/p^n})|\mathbb{Q}_p$ is étale but the extension $\mathbb{Z}_p[p^{1/p^n}]|\mathbb{Z}_p$ is ramified. Hence the perfectoid K-algebra $K\langle X^{\pm 1/p^\infty}\rangle$ arises from a tower of K-algebras that is "infinitely ramified on the mod p fibre", like in our examples of perfectoid fields in §1.1.1.6. We will further explore in Lecture 3 how perfectoid spaces arise as infinitely ramified towers.

Another source of perfectoid algebras stems from perfection, generalising §1.1.1.5:

Proposition 1.2.12. *Assume* $\mathrm{Char}(K) = p$ *and let* (R, R^+) *be a Huber pair over* K. *Consider* $\varinjlim_F R^+$. *Then the* ϖ*-adic completion* $R^{+,\mathrm{perf}}$ *of* $\varinjlim_F R^+$ *is a perfect* $\mathcal{O}_K$*-algebra. In particular,* $R^{\mathrm{perf}}[\frac{1}{\varpi}] := R^{+,\mathrm{perf}}[\frac{1}{\varpi}]$ *is a perfectoid* K*-algebra.*

Exercise 1.2.13. Prove this. Hint: Exercise 1.2.6 helps.

1.2.1.2 Perfectoid $\mathcal{O}_K$-algebras. Let K be a perfectoid field. As before, we fix a pseudo-uniformizer $\varpi \in K$ that admits arbitrary p-th roots and such that $\varpi | p$.

As we have seen in the examples, when we want to check that a Huber ring R over K is perfectoid, it can be difficult in practice to verify the condition that R is uniform. Similarly, it is often difficult to compute the ring R°/p explicitly in order to verify that the Frobenius is surjective. This is one reason why the following is a very useful criterion to see that algebras are perfectoid:

Proposition 1.2.14 ([Sch12, Lemma 5.6]). *Let* A *be an* $\mathcal{O}_K$*-algebra that is* ϖ*-adically complete,* ϖ*-torsionfree and such that*

$$\Phi : A/\varpi^{1/p} \to A/\varpi, \quad x \mapsto x^p$$

is an isomorphism. Then $A[\frac{1}{\varpi}]$ *is a perfectoid* K*-algebra.*

Definition 1.2.15. We call such A perfectoid $\mathcal{O}_K$-algebras.

The following proof of Proposition 1.2.14 is due to Scholze. We modify it a bit to make it slightly more elementary, by removing some "almost mathematics".

Proof. Let $R := A[\frac{1}{\varpi}]$. We claim that we have an explicit description of R° as follows:

$$A_* := \{x \in R \mid \text{for all } n \in \mathbb{N}\colon \varpi^{1/p^n} \cdot x \in A\} = R^\circ. \tag{1.3}$$

We first observe that $A_* \subseteq R^\circ$: Indeed, for any $x \in A_*$, we have $\varpi \cdot x^{p^n} \in A$ for all n, hence x is power-bounded. For the other direction, we use:

Claim 1.2.16 ([Sch12, Lemma 5.7]). *Let* $x \in R$ *such that* $x^p \in A_*$. *Then* $x \in A_*$.

Proof. Fix $n \in \mathbb{N}$ and set $\epsilon := \varpi^{\frac{1}{p^n}}$. We need to see that $\epsilon x \in A$. Let $k \geq 1$ be large enough such that $y_0 := \varpi^{\frac{k}{p}} x \in A$. Then $y_0^p = \varpi^k x^p \in \varpi A_*$. Hence $(\epsilon y_0)^p \in \varpi A$ for any $n \in \mathbb{N}$. Since Φ is injective, it follows that

$$\epsilon \varpi^{\frac{k}{p}} x = \epsilon y_0 \in \varpi^{\frac{1}{p}} A.$$

Since A has no ϖ-torsion, we can cancel a factor of $\varpi^{\frac{1}{p}}$ from this equation. It follows that $\epsilon \varpi^{\frac{k-1}{p}} x \in A$. This shows that $y_1 := \varpi^{\frac{k-1}{p}} x \in A_*$. Continuing inductively, we see that $y_k = x \in A_*$, as we wanted to see. ■

Let now $x \in R^\circ$. For any $n \in \mathbb{N}$, we need to see that $\epsilon \cdot x \in A$ where $\epsilon = \varpi^{1/p^n}$. To this end, observe first that $(\epsilon \cdot x)^k \to 0$ for $k \to \infty$ as x is power-bounded and $\epsilon^k \to 0$. Since $A \subseteq A[\frac{1}{\varpi}]$ is open, this shows that there is $k \in \mathbb{N}$ such that $(\epsilon \cdot x)^{p^k} \in A$. The claim now shows that $\epsilon \cdot x \in A$, as desired. This completes the proof of (1.3).

From (1.3), we deduce that $\varpi \cdot R^\circ \subseteq A$. Hence R is uniform.

Finally, we use (1.3) to verify that $R^\circ/\varpi^{1/p} \to R^\circ/\varpi$, $x \mapsto x^p$ is surjective. In fact, by Exercise 1.2.2, we are free to change ϖ, so it suffices to see that

$$\Phi' : R^\circ/\varpi^{1/p^2} \to R^\circ/\varpi^{1/p}, \quad x \mapsto x^p$$

is surjective. To verify this, let $x \in R^\circ$. Then we have $\varpi^{1/p}x \in A$, this is the case of $n = 1$ of the condition in (1.3). Since Φ is surjective by assumption, there is $y \in A$ such that $y^p = \varpi^{1/p}x + \varpi \cdot r$ for some $r \in A$. Then $z := \varpi^{-1/p^2}y \in R$ satisfies

$$z^p = x + \varpi^{1-\frac{1}{p}}r \in R^\circ.$$

Since R° is integrally closed, this implies that $z \in R^\circ$. Hence this equation shows that the residue class of x is in the image of Φ', as we wanted to see. ∎

Remark 1.2.17. The arguments involving factors of ϖ^{1/p^n} for any n give a first glimpse at (and motivation for) "almost mathematics", which we meet again in §1.2.2.1.

Exercise 1.2.18. Revisit the examples in Section 1.2.1.1 and convince yourself that Proposition 1.2.14 makes it much easier to verify that these are perfectoid.

Exercise 1.2.19.

(1) For a perfectoid K-algebra R, any ring of integral elements $R^+ \subseteq R$ is a perfectoid $\mathcal{O}_K$-algebra. Hint: Show that $\omega^{1/p}R^\circ \subseteq R^+$ and use that you may change the uniformizer ϖ.

(2) A concrete example: Let $R_1 := \mathcal{O}_K\langle X^{1/p^\infty}\rangle$ and $R_2 := \mathcal{O}_K + \mathfrak{m}_K\mathcal{O}_K\langle X^{1/p^\infty}\rangle$. Verify that these both define perfectoid $\mathcal{O}_K$-subalgebras of $K\langle X^{1/p^\infty}\rangle$.

1.2.1.3 The Tilting equivalence. It is straightforward to adapt the basic "perfectoid" constructions for perfectoid fields from Lecture 1 to this setup, as we now explain.

From now on, let us fix a perfectoid field K with tilt $K^\flat$, let $\varpi^\flat$ be a pseudo-uniformizer of $K^\flat$ with $\varpi|p$ and let $\varpi = \varpi^{\flat\sharp}$. We start with the tilting functor:

Definition 1.2.20. For any perfectoid K-algebra R, its **tilt** is the $K^\flat$-algebra $R^\flat$ defined as

$$R^{\circ\flat} := \varprojlim_F R^\circ/p, \quad R^\flat := R^{\circ\flat}[\tfrac{1}{\varpi^\flat}].$$

We will see in Exercise 1.2.23 below that $R^{\circ\flat} = R^{\flat\circ}$, hence we will later use these two interchangeably.

Exercise 1.2.21.

(1) Show that $R^\flat$ is a perfectoid $K^\flat$-algebra. Hint: Exercise 1.2.6.

(2) Show that $\varprojlim_F R^\circ/p = \varprojlim_F R^\circ/\varpi$, so we could equivalently use R°/ϖ to define $R^\flat$. Since we always allow the case that $\mathrm{Char}(K) = p$, this shows in particular that in characteristic p, we have $R^\circ = \varprojlim_F R^\circ/\varpi$.

(3) Show that there is a natural isomorphism $R^{\circ\flat}/\varpi^\flat = R^\circ/\varpi$.

(4) Show that for the perfectoid algebra from Proposition 1.2.8, there is a canonical isomorphism $K\langle X^{1/p^\infty}\rangle^\flat = K^\flat\langle X'^{1/p^\infty}\rangle$ where X' is defined as the element $(X, X^{1/p}, X^{1/p^2}, \dots) \in \varprojlim_F \mathcal{O}_K\langle X^{1/p^\infty}\rangle/p$.

Exactly as in the case of perfectoid fields, one shows:

Proposition 1.2.22. *The natural map of sets*

$$\varprojlim_{x\mapsto x^p} R^\circ \to \varprojlim_{x\mapsto x^p} R^\circ/p$$

is a bijection. In particular, there is a multiplicative bijection $R^\flat = \varprojlim_{x\mapsto x^p} R$.

We deduce that the projection to the first component of $\varprojlim_{x\mapsto x^p} R$ is a multiplicative map, the sharp map

$$\sharp : R^\flat \to R$$

which turns out to be continuous, like we saw for perfectoid fields.

Second, we again have Fontaine's θ-map which is defined by

$$\theta : W(R^{\flat\circ}) \to R^\circ, \qquad \sum_{n=0}^{\infty} [a_n]p^n \mapsto \sum_{n=0}^{\infty} a_n^\sharp p^n.$$

Once again, it is a non-trivial result that this is a $\mathbb{Z}_p$-algebra homomorphism.

We can use the $\sharp$-map to see:

Exercise 1.2.23. Prove that for the ring of power-bounded elements $R^{\flat\circ}$ of $R^\flat$, we have

$$R^{\flat\circ} = R^{\circ\flat}.$$

Hint: Convince yourself that the inclusion $R^{\circ\flat} \subseteq R^{\flat\circ}$ is clear. For the other inclusion, let $f \in R^{\flat\circ}$, then there is n such that $\varpi^{\flat n} f^m \in R^{\circ\flat}$ for every $m \in \mathbb{N}$. Now apply $\sharp$. If you are stuck, consult [Bha17, Theorem 6.2.7.3].

1.2.1.4 The Tilting equivalence for perfectoid algebras. Having discussed the tilting functor, our next goal is to see that one can again go into the opposite direction, from characteristic p to characteristic 0. For this we generalize Definition 1.1.38 from perfectoid fields to perfectoid algebras:

Definition 1.2.24. For any perfectoid $K^\flat$-algebra S, we define its untilt $S^\sharp$ to be

$$S^{\sharp\circ} := W(S^\circ) \otimes_{W(\mathcal{O}_{K^\flat})} \mathcal{O}_K, \qquad S^\sharp := S^{\sharp\circ}[\tfrac{1}{\varpi}].$$

We can now state the first main result of this lecture, generalising Theorem 1.1.39:

Theorem 1.2.25 (Tilting equivalence). *Tilting and untilting define equivalences*

$$\{\textit{perfectoid } K\textit{-algebras}\} \xrightarrow{\sim} \{\textit{perfectoid } K^\flat\textit{-algebras}\}, \quad R \mapsto R^\flat, \quad S^\sharp \leftarrow\!\shortmid S.$$

Note that we make no assumption that $\mathrm{Char}\, K = 0$ (although this is the main case of interest). In fact, the statement also holds in characteristic p. Then $R^\flat = R$.

Proof. Let $\varpi \in O_K$ and $\varpi^\flat \in O_{K^\flat}$ be as before. Then by Exercise 1.2.21, we have

$$S^{\sharp\flat\circ} = \varprojlim_F (W(S^\circ) \otimes_{W(O_{K^\flat})} O_K)/\varpi = \varprojlim_F S^\circ \otimes_{O_{K^\flat}} O_K/\varpi = \varprojlim_F S^\circ/\varpi^\flat = S^\circ$$

where we have used that $\ker(O_{K^\flat} \to O_K/\varpi)$ is generated by $\varpi^\flat$.

Recall from Proposition 1.1.35 that $W(O_{K^\flat}) \to O_K$ is surjective and its kernel is generated by a single element $\xi \in W(O_{K^\flat})$. We can therefore identify

$$W(S^\circ) \otimes_{W(O_{K^\flat})} O_K \cong W(S^\circ)/\xi.$$

To see that $R^{\flat\sharp} = R$, observe that by functoriality of θ, there is a natural map

$$R^{\flat\sharp\circ} = W(R^{\flat\circ}) \otimes_{W(O_{K^\flat})} O_K \xrightarrow{\theta} R^\circ.$$

If $\mathrm{Char}(K) = p$, this is clearly an isomorphism. Assume now that $\mathrm{Char}(K) = 0$. Then by the same argument as in the first part of the proof, this map is an isomorphism mod ϖ. To conclude that the map is is an isomorphism, it therefore suffices to note that both sides are ϖ-adically complete: Indeed, ϖ is a non-zero divisor on $W(O_{K^\flat})/\xi \cong O_K$, hence ξ is a non-zero divisor mod ϖ. It follows inductively that for any n, the sequence

$$0 \to W(R^{\flat\circ})/\varpi^n \xrightarrow{\cdot\xi} W(R^{\flat\circ})/\varpi^n \to W(R^{\flat\circ})/(\xi, \varpi^n) \to 0$$

is exact. Applying $\varprojlim_{n\in\mathbb{N}}$, this shows that $W(R^{\flat\circ})/\xi$ is ϖ-adically complete. ∎

Remark 1.2.26. Let R be any perfectoid K-algebra. Note that the definition of the tilt $R^\flat$ only depends on R°/p. We can therefore reconstruct R from R°/p, as follows:

Corollary 1.2.27. *There is a natural isomorphism*

$$W(\varprojlim_F R^\circ/p) \otimes_{W(O_{K^\flat}),\theta} O_K \xrightarrow{\sim} R^\circ.$$

That such a reconstruction is possible is very surprising! One would usually expect the passage from R° to R°/p to loose a lot of information. The phenomenon that we can recover information about the generic fibre R from the mod-p fibre R°/p is quite a characteristical property of perfectoid algebras that we again encounter later.

Remark 1.2.28. Note that when $\mathrm{Char}(K) = p$, Corollary 1.2.27 simplifies to the statement from Exercise 1.2.21 that $R^\circ = \varprojlim_F R^\circ/\varpi$. Contemplating this identity gives a good first intuition for why such perfectoid "reconstruction" results are possible.

Remark 1.2.29. We can now give some indication why uniformity is required in Definition 1.2.1: Consider $R = K\langle X\rangle/X^2$. This is a K-Banach algebra for which R° contains the whole line KX. Hence R° is not bounded, so R is not uniform. In fact, $R^\circ = \mathcal{O}_K + KX$ and thus $R^\circ/p = \mathcal{O}_K/p$. So $F : R^\circ/p \to R^\circ/p$ is surjective, but in some sense for the "wrong" reason! Note for example that R°/p is too small to be able to reconstruct R° from R°/p, so we certainly cannot expect phenomena like Corollary 1.2.27 when we drop the uniformity condition.

1.2.1.5 Aside: Alternative notions of perfectoid rings. The definition of perfectoid rings has varied a bit over time. The reason is simply that they were quickly used very widely, and it was desirable to modify the definition slightly for different purposes:

- Recall that by the definition we follow in this course, a perfectoid algebra always lives over a perfectoid field. There are other, newer definitions of perfectoid spaces that do not require this assumption. In [SW20, Definition 6.1.1], a **perfectoid ring** is instead defined to be a complete Tate ring R that is uniform and such that there is a pseudo-uniformizer π with $\pi^p|p$ in R° for which $F : R^\circ/\pi \to R^\circ/\pi^p$ is an isomorphism. With this definition, there are examples of perfectoid rings that do not even assume that R contains *any* field!
- There is also an integral variant of "perfectoid rings": For example, in [BMS18, Definition 3.5], a "perfectoid ring" is defined to be a ring S which is π-adically complete for some element $\pi \in S$ such that $\pi^p|p$, and for which $F : S/\pi \to S/\pi^p$ is an isomorphism. An example would be $S = R^\circ$ for a perfectoid K-algebra R. In other works, this notion is also sometimes called an **integrally perfectoid ring** to distinguish it from the "analytic" version we consider in these lectures. But there is currently no single naming convention that is consistently used in the literature.
- There are equivalent integral definitions which are also in use, for example one using Fontaine's θ-map [Fon13][BMS18, §3], or in terms of perfect prisms [BS].

Upshot: If one encounters a "perfectoid ring", this can mean different things in different contexts. In practice, this is usually not an issue (and a slight ambiguity in naming conventions that is clarified in the beginning of an article is quite common, e.g. a "rigid space" over K might be a rigid analytic space over K in the sense of Tate or an adic space of locally topologically finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Huber).

1.2.2 Almost Purity

1.2.2.1 The formalism of "almost mathematics". The proof of Theorem 1.2.25 that Scholze gives in [Sch13a] is different to the one given above, and yields stronger results. In order to be able to formulate these, we need more machinery:

The first crucial ingredient is the theory of **almost mathematics**. This was introduced by Faltings and developed in detail by Gabber–Ramero [GR03]. For some concrete motivation why this notion appears in this context, see §1.2.2.2 below.

Throughout, K is a perfectoid field with uniformizer ϖ admitting p-power roots.

Definition 1.2.30. (1) Let M be an $\mathcal{O}_K$-module. An element $m \in M$ is said to be **almost zero** if for every $x \in \mathfrak{m}_K$, we have $x \cdot m = 0$. The module M is said to be almost zero if every $m \in M$ is almost zero. Equivalently, this means that

$$M \otimes_{\mathcal{O}_K} \mathfrak{m}_K = 0.$$

A morphism of $\mathcal{O}_K$-modules is said to be an **almost isomorphism** if it's kernel and cokernel are almost zero.

(2) The category of $\mathcal{O}_K^a$-modules is the category of $\mathcal{O}_K$-modules localised at the almost isomorphisms (i.e. it is the quotient of the category of $\mathcal{O}_K$-modules by the almost zero modules). For any $\mathcal{O}_K$-module M, we denote by M^a its image in the almost category. To distinguish between the almost category and the usual one, we shall also call M an "honest $\mathcal{O}_K$-module" and in contrast call M^a the corresponding "**almost $\mathcal{O}_K$-module**".

(3) For any $\mathcal{O}_K$-modules M, N with a given map $f : M \to N$, we write $M \overset{a}{=} N$ if f induces an almost isomorphism between M and N (the "a" is for "almost"). In particular, $M \overset{a}{=} 0$ means that M is annihilated by any element in $\mathfrak{m}_K$.

The book [GR03] develops a whole theory of commutative algebra in the almost category. A summary that is sufficient for most perfectoid applications is given in [Sch12, §4]: There is a natural tensor product making $\mathcal{O}_K^a$-modules an abelian tensor category. In particular, there is a notion of an $\mathcal{O}_K^a$-algebra: This is a commutative algebra object in $\mathcal{O}_K^a$-modules, or equivalently it is of the form A^a for an $\mathcal{O}_K$-algebra A [GR03, Proposition 2.2.13]. As inverting ϖ kills ϖ-torsion modules, there is a "generic fibre" functor $R \mapsto R[\frac{1}{\varpi}]$ from $\mathcal{O}_K^a$-algebras to K-algebras.

Using the natural tensor product, one can define a notion of flat $\mathcal{O}_K^a$-algebras.

Exercise 1.2.31. Let R be a Huber ring over K with ring of integral elements $R^+ \subseteq R^\circ$. Then $R^+ \overset{a}{=} R^\circ$. Hint: Use that any $x \in \mathfrak{m}_K R^\circ$ is topologically nilpotent.

Second, we have already encountered an almost isomorphism during the proof of Proposition 1.2.14. Indeed, we can now interpret (1.3) as saying:

Corollary 1.2.32. *If A is a perfectoid $\mathcal{O}_K$-algebra, then $A \overset{a}{=} A[\frac{1}{\varpi}]^\circ$.*

Scholze now introduces the following "almost" version of Definition 1.2.15:

Definition 1.2.33. (1) Let R be a flat $\mathcal{O}_K^a$-algebra. Like in Exercise 1.2.2, we can define a morphism of $\mathcal{O}_K^a$-algebras

$$\Phi : R/\varpi^{1/p} \to R/\varpi, \quad x \mapsto x^p$$

by considering the target as an $\mathcal{O}_K^a$-algebra via $F : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi$. We call R perfectoid if it is ϖ-adically complete and Φ is an almost isomorphism.

(2) Let R be a flat $(\mathcal{O}_K/\varpi)^a$-algebra. In the same way as in (1), we can define a morphism $\Phi : R/\varpi^{1/p} \to R$, $x \mapsto x^p$ that is linear over $(\mathcal{O}_K/\varpi^{1/p})^a$, where we endow the target with an algebra structure via $F : \mathcal{O}_K/\varpi^{1/p} \xrightarrow{\sim} \mathcal{O}_K/\varpi$. We call R perfectoid if Φ is an isomorphism.

Here the conditions "ϖ-adically complete" and "flat" are understood to be in terms of almost mathematics. But there is a close relation to perfectoid $\mathcal{O}_K$-algebras:

Lemma 1.2.34 ([Sch12, Lemma 5.3, Lemma 5.6]). *An $\mathcal{O}_K^a$-algebra R is perfectoid if and only if $R' := R[\frac{1}{\varpi}]^\circ$ is a perfectoid $\mathcal{O}_K$-algebra in the sense of Definition 1.2.15.*

An advantage of the almost setting is that we have the following equivalence:

Proposition 1.2.35. *We have an equivalence of categories*

$$\mathrm{Perf}_K \to \mathrm{Perf}_{\mathcal{O}_K^a}, \quad R \mapsto R^\circ, \quad A[\tfrac{1}{\varpi}] \leftarrow\!\shortmid A.$$

Proof. We need to check that the map $A \to A[\frac{1}{\varpi}]^\circ$ is an almost isomorphism. For this one can argue like in the proof of (1.3). See [Sch12, Lemma 5.6] for details. ∎

Remark 1.2.36. We can write down the same functors for $\mathrm{Perf}_{\mathcal{O}_K}$ instead of $\mathrm{Perf}_{\mathcal{O}_K^a}$. But they would not be an equivalence of categories! Indeed, by Exercise 1.2.19, any ring of integral elements $R^+ \subseteq R^\circ$ is a perfectoid $\mathcal{O}_K$-algebras such that $R^+[\frac{1}{\varpi}] = R$.

But $R^+ \overset{a}{=} R^\circ$ by Exercise 1.2.31, hence the two become isomorphic in $\mathrm{Perf}_{\mathcal{O}_K^a}$. You can try this out at the example from Exercise 1.2.19.(2).

Second, there is clearly a reduction functor

$$\mathrm{Perf}_{\mathcal{O}_K^a} \to \mathrm{Perf}_{(\mathcal{O}_K/\varpi)^a}, \quad A \mapsto A/\varpi.$$

The crucial and surprising point is that this is an equivalence. In fact, Scholze proves:

Theorem 1.2.37 ([Sch12, Theorem 5.2]). *We have a commutative diagram of equivalences*

$$\begin{array}{ccccc}
\mathrm{Perf}_K & \xrightarrow[\sim]{R\mapsto R^\circ} & \mathrm{Perf}_{\mathcal{O}_K^a} & \xrightarrow[\sim]{A\mapsto A/\varpi} & \mathrm{Perf}_{\mathcal{O}_K^a/\varpi} \\
\flat \downarrow \wr & & & & \downarrow \wr \\
\mathrm{Perf}_{K^\flat} & \xrightarrow[\sim]{R\mapsto R^\circ} & \mathrm{Perf}_{\mathcal{O}_{K^\flat}^a} & \xrightarrow[\sim]{A\mapsto A/\varpi^\flat} & \mathrm{Perf}_{\mathcal{O}_{K^\flat}^a/\varpi^\flat}
\end{array}$$

where the right vertical functor is induced by the isomorphism $\mathcal{O}_K^a/\varpi \cong \mathcal{O}_{K^\flat}^a/\varpi^\flat$.

This also gives a second proof of Theorem 1.2.25. The key step is proving that any perfectoid $\mathcal{O}_K^a/\varpi$-algebra has a unique lift to a perfectoid $\mathcal{O}_K^a$-algebra. This can be seen using an almost version of the cotangent complex, which on the one hand classifies a lifting obstruction, and on the other hand vanishes for perfect algebras.

1.2.2.2 Almost étale ring maps: An illustrative example. Our next goal is to show that the equivalences in Theorem 1.2.37 match up étale morphisms:

Let A be an $\mathcal{O}_K^a$-algebra. There is a notion of an **étale A-algebra in the almost category**. The definition is quite technical as it relies on a lot of "almost commutative algebra". We briefly summarise the main definitions here in the notes to give a flavour of it and refer to [Sch12, §4] for details: Let R be any $\mathcal{O}_K$-algebra such that $A = R^a$.

- An A-module M is flat when $-\otimes_R M$ is exact on the category of R^a-modules.
- An A-module N is almost finitely presented if there is an R-module M with $N = M^a$ for which there is for any $\epsilon \in \mathfrak{m}_K$ a finitely presented R-algebra N_ϵ with a morphism $N_\epsilon \to N$ that has ϵ-torsion kernel and cokernel.
- Finally, there is a notion of an unramified A-algebra, defined in terms of "almost elements". An A-algebra is called étale if it is flat and unramified. It is called finite étale if it is étale and almost finitely presented. We denote by $A_{\text{fét}}$ the category of finite étale A-algebras.

That is quite a bunch of technical definitions. In order to get a feeling for what this definition captures, it is perhaps most helpful to look at an illustrative example:

Example 1.2.38. Assume $p \neq 2$ and let $K = \mathbb{F}_p((t^{1/p^\infty}))$. Consider the finite field extension $L := K(t^{1/2})|K$. This is generated by the element $t^{1/2}$ which has the characteristic polynomial $X^2 - t$, hence $[L : K] = 2$. We can think of $L|K$ as the perfection of any of the finite extensions

$$L_n|K_n, \quad \text{where } K_n := \mathbb{F}_p((t^{1/p^n})), \quad L_n := K_n(t^{1/2}).$$

These are finite extensions of local fields. A uniformizer of L_n is given by $\pi_n := t^{\frac{1}{2p^n}}$ (Exercise: check this!). The associated extension of rings of integers is

$$\mathcal{O}_{L_n}|\mathcal{O}_{K_n} \quad \text{where } \mathcal{O}_{K_n} = K_n^\circ = \mathbb{F}_p[[t^{\frac{1}{p^n}}]], \quad \mathcal{O}_{L_n} = L_n^\circ = \mathcal{O}_{K_n}[\pi_n].$$

The minimal polynomial of π_n is $E(X) = X^2 - t^{1/p^n}$. Recall now that there is a quantity measuring the ramification in the extension $\mathcal{O}_{L_n}|\mathcal{O}_{K_n}$, called the different ideal:

$$\delta_{L_n|K_n} = E'(\pi_n)\mathcal{O}_{L_n} = t^{\frac{1}{2p^n}}\mathcal{O}_{L_n}.$$

What do we mean by "measuring"? For an unramified extension the different is $\mathcal{O}_{L_n}$. So we can think of the size of the $\mathcal{O}_{L_n}$-module $\mathcal{O}_{L_n}/\delta_{L_n|K_n}$ as measuring ramification.

We now take the limit over n: Note that as $n \to \infty$, the t-adic valuation of $\delta_{L_n|K_n}$ is $\frac{1}{2p^n} \to 0$, meaning that the extension becomes "less and less ramified".

Exercise 1.2.39. (1) Prove that the extension $\mathcal{O}_L|\mathcal{O}_K$ is no longer finite! Hence finite extensions of perfectoid fields behave quite a bit differently to what we are used to from local fields. Hint: That $t^{\frac{1}{2p^n}} \in \mathcal{O}_{L_n}$ for all n can be seen by the Euclidean algorithm, namely there are $j, k \in \mathbb{Z}$ such that $\frac{j}{p^n} + \frac{k}{2} = \frac{1}{2p^n}$. However, we cannot take $j, k \geq 0$.

(2) Convince yourself that one can still make sense of the different ideal $\delta_{L|K}$ using the trace $\mathrm{Tr}_{L|K}$ of the finite extension $L|K$. Use this to show $\delta_{L|K} = \mathfrak{m}_L$.

This illustrates in what sense the extension $\mathcal{O}_L|\mathcal{O}_K$ is "almost finite étale": It is approximated by the almost finitely presented extensions $\mathcal{O}_K[\pi_n]|\mathcal{O}_K$ ($\Rightarrow$ almost finitely presented), and the quotient module $\mathcal{O}_L/\delta_{L|K}$ measuring its ramification is almost zero ($\Rightarrow$ almost unramified). But in terms of "honest" rather than almost commutative algebra, it is neither finite nor étale.

Exercise 1.2.40. Compute the different ideal of the almost finite étale extension $\mathcal{O}_L|\mathcal{O}_K$ where $K := \mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $L := K(p^{1/n})$ is its finite extension defined by $X^n - p$ where $n \in \mathbb{N}$ is coprime to p. This is a characteristic 0 analogue to the above example.

Phenomena as discussed above were well-known early on in p-adic geometry, see e.g. [Tat67, §3], or [BGR84, §3.6.1, §6.4.1] where the fact that $\mathcal{O}_L|\mathcal{O}_K$ need not be finite is discussed at the example of what we today call the perfectoid field $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$.

1.2.2.3 Statement of Almost Purity. In §1.2.2.2, we have just seen a finite étale extension $L|K$ of perfectoid fields which gives rise to an almost finite étale extension $\mathcal{O}_L|\mathcal{O}_K$. It turns out that this works much more generally for perfectoid K-algebras:

Theorem 1.2.41 ([Sch12, Theorems 1.10 and 7.9]). *Let K be a perfectoid field and R a perfectoid K-algebra. The equivalences of Theorem 1.2.37 restrict to equivalences*

$$\begin{array}{ccccc}
R_{\mathrm{f\acute{e}t}} & \xrightarrow[\sim]{R\mapsto R^\circ} & R^{\circ a}_{\mathrm{f\acute{e}t}} & \xrightarrow[\sim]{A\mapsto A/\varpi} & (R^{\circ a}/\varpi)_{\mathrm{f\acute{e}t}} \\
 & & & & \downarrow\wr \\
R^\flat_{\mathrm{f\acute{e}t}} & \xrightarrow[\sim]{R\mapsto R^\circ} & R^{\flat\circ a}_{\mathrm{f\acute{e}t}} & \xrightarrow[\sim]{A\mapsto A/\varpi^\flat} & (R^{\flat\circ a}/\varpi^\flat)_{\mathrm{f\acute{e}t}}.
\end{array}$$

That $A \mapsto A/\varpi$ is an equivalence follows from the almost version of Henselian lifting. This time, proving that $R \mapsto R^\circ$ is an equivalence is where the main work is. This is one of the main results of [Sch12]. The statement can be reformulated as follows:

Theorem 1.2.42 (Almost purity). *Let R be a perfectoid K-algebra. Then for any finite étale ring extension $S|R$, the extension $S^\circ|R^\circ$ is almost finite étale.*

This generalises a result of Faltings [Fal02, 4. Theorem]. For perfectoid fields, it can be proved using earlier arguments by Tate [Tat67] and Gabber–Ramero [GR03].

The basic idea for Scholze's proof of Theorem 1.2.42 is to first prove the statement in characteristic p, where ramification can be made arbitrarily small by iterating the Frobenius. He then deduces the result in characteristic 0 by tilting. This is hard: He uses a geometric argument in terms of perfectoid spaces to eventually reduce to perfectoid fields.

Corollary 1.2.43. *Let $R \to S$ be a finite étale map where R is a perfectoid K-algebra. Then S is a perfectoid K-algebra. Tilting thus induces an equivalence of sites*

$$-^\flat : R_{\text{fét}} \xrightarrow{\sim} R^\flat_{\text{fét}}.$$

Proof. As S is finite projective over R, it inherits a natural K-Banach algebra structure from R. In characteristic p, the statement now follows from Exercise 1.2.44 below combined with Exercise 1.2.6. The result in general follows from Theorem 1.2.41 via untilting, i.e. comparing the diagram to that in Theorem 1.2.37. ∎

Exercise 1.2.44. Let R be a perfect $\mathbb{F}_p$-algebra. Let $R \to S$ be étale. Show that S is still perfect. Hint: Show that the absolute Frobenius morphism $F : \mathrm{Spec}(S) \to \mathrm{Spec}(S)$ is étale and a universal homeomorphism. This implies that it is an isomorphism.

Remark 1.2.45. Since it is the most difficult step in the proof, many authors just say "by almost purity" when they invoke Theorem 1.2.41 or Corollary 1.2.43.

1.3 Perfectoid spaces

1.3.1 Affinoid perfectoid spaces

Having discussed perfectoid K-algebras, we now pass from algebras to spaces.

Definition 1.3.1. An **affinoid perfectoid space** over K is an adic space of the form

$$X = \mathrm{Spa}(R, R^+)$$

for a perfectoid K-algebra R and any ring of integral elements $R^+ \subseteq R^\circ$. Recall from the course on adic spaces that this means that R^+ is open and integrally closed in R.

Remark 1.3.2. At first reading, it is ok not to worry too much about R^+. We can always take $R^+ = R^\circ$. In fact, by Exercise 1.2.31, this is "almost the only choice".

That all being said, the greater generality of R^+ is useful in general. An example for a ring of integral elements $R^+ \subset R^\circ$ with $R^+ \neq R^\circ$ is described in Exercise 1.2.19.

1.3.1.1 Rational subspaces. The first goal of this section is to show that any affinoid perfectoid space is a sheafy adic space. This is not easy to see. We will deduce it from a different result which is in itself very interesting:

Theorem 1.3.3 ([Sch12, Theorem 6.3]). *Any rational subspace of an affinoid perfectoid space is again an affinoid perfectoid space.*

This means that for any $f_1, \dots, f_r \in R$ generating the unit ideal in R and any $g \in R$, the rational localisation $R\langle \frac{f_1}{g}, \dots, \frac{f_r}{g} \rangle$ is automatically a perfectoid K-algebra.

Exercise 1.3.4. Prove this by hand in the case of $R = K\langle T^{1/p^\infty} \rangle$, $f_1 = 1$, $g = T$.

Corollary 1.3.5. *Any affinoid perfectoid space is a sheafy adic space.*

Proof. As perfectoid rings are uniform, Theorem 1.3.3 implies that affinoid perfectoid spaces are stably uniform. The result now follows from the Theorem of Mihara and Buzzard–Verberkmoes ("stably uniform affinoids are sheafy") [Mih16][BV18] ■

We give a sketch of the proof of Theorem 1.3.3: Once again, the idea is to first prove the statement when $\mathrm{Char}(K) = p$, and deduce the case of $\mathrm{Char}(K) = 0$ by untilting. For this we first need to make sense of "tilting and untilting rational subspaces".

As the very first step, we observe that tilting identifies integral subrings:

Lemma 1.3.6 ([Sch12, Lemma 6.2]). *Sending $R^+ \mapsto R^{+\flat} := \varprojlim_F R^+/p$ defines a bijection*

$$\{\textit{subrings of integral elements of } R^\circ\} \xrightarrow{\sim} \{\textit{subrings of integral elements of } R^{\flat\circ}\}.$$

Proof. By Exercise 1.3.7 below, subrings of integral elements of R° correspond bijectively to integrally closed subrings of R°/ϖ. The same description applies to $R^{\flat\circ}$ and $R^{\flat\circ}/\varpi$. Now we use that $R^\circ/\varpi \cong R^{\flat\circ}/\varpi$. ■

Exercise 1.3.7. Show that sending $A \mapsto A/\varpi$ defines a bijection between the integrally closed open subrings $A \subseteq R^\circ$ and the integrally closed subrings of R°/ϖ. Hint: Show first that $\varpi R^\circ \subseteq R^\circ$ is contained in any integrally closed open subring of R°.

Definition 1.3.8. For any affinoid perfectoid space $X = \mathrm{Spa}(R, R^+)$, we set

$$X^\flat := \mathrm{Spa}(R^\flat, R^{+\flat}).$$

This is evidently an affinoid perfectoid space over $K^\flat$. The next aim is to define a morphism of topological spaces $|X| \to |X^\flat|$.

Lemma 1.3.9. *For any valuation $v : R \to \Gamma$, the map of sets $v^\flat : R^\flat \xrightarrow{\sharp} R \xrightarrow{v} \Gamma$ is a valuation of $R^\flat$. Here $\sharp : R^\flat \to R$ is the sharp map from Section 1.2.1.3.*

Exercise 1.3.10. Prove this! Hint: The main work lies in proving that $v^\flat(a + b) \geq \min(v^\flat(a), v^\flat(b))$. For this, use that $(a + b)^\sharp = \lim_n (a^{1/p^n \sharp} + b^{1/p^n \sharp})^{p^n}$.

We can use this to define a map of sets

$$-^\flat : |X| = |\operatorname{Spa}(R, R^+)| \to |X^\flat| = |\operatorname{Spa}(R^\flat, R^{+\flat})|, \quad v \mapsto v^\flat.$$

For any open subspace $U \subseteq X$, we denote its image under this map by $U^\flat \subseteq X^\flat$.

Theorem 1.3.11. *The map $-^\flat : |X| \to |X^\flat|$ is a homeomorphism which identifies the rational subspaces on both sides. For any rational open $U \subseteq X$, we have isomorphisms*

$$\mathcal{O}_X(U)^\flat = \mathcal{O}_{X^\flat}(U^\flat), \quad \mathcal{O}_X^+(U)^\flat = \mathcal{O}_{X^\flat}^+(U^\flat).$$

Proof of Theorems 1.3.3 and 1.3.11. The proof of these results is a bit involved and intertwined. It is completed in several steps that we now sketch, following [Sch12, §6].

Step 1: For any rational open $U := X^\flat(\frac{f_1,\dots,f_r}{g}) \subseteq X^\flat$, set $U^\sharp := X(\frac{f_1^\sharp,\dots,f_r^\sharp}{g^\sharp})$. Then one can verify by hand that $U^\sharp$ is the preimage of U under $-^\flat$. This shows that rational opens pull back to rational opens, hence $-^\flat$ is continuous.

Step 2: One proves that U is affinoid perfectoid, i.e. Theorem 1.3.3 holds in characteristic p. To simplify notation, let us treat the case of $r = 1$, the general case is entirely analogous but requires more notation: We consider the ring

$$S_0 := R^+\langle (\tfrac{f}{g})^{\frac{1}{p^\infty}} \rangle := R^+\langle T^{1/p^\infty} \rangle / I$$

where I is the closure of the ideal $(f^{1/p^n} - T^{1/p^n} g^{1/p^n} | n \in \mathbb{N})$. Then we have:

Exercise 1.3.12. S_0 is a perfectoid $\mathcal{O}_K^a$-algebra.

It follows from Theorem 1.2.37 that $S_0[\frac{1}{\varpi}]$ is a perfectoid K-algebra.

Step 3: Using Theorem 1.2.37, an explicit computation of the algebras mod ϖ shows

$$\mathcal{O}_X(U^\sharp) = \mathcal{O}_{X^\flat}(U)^\sharp.$$

As $\mathcal{O}_{X^\flat}(U)^\sharp$ is a perfectoid K-algebra, and as we know from Step 1 that $U^\sharp$ is affinoid (it is rational open in an affinoid adic space), this shows that $U^\sharp$ is affinoid perfectoid.

Step 4: We show $-^\flat : |X| \to |X^\flat|$ is surjective: This relies on the following lemma.

Lemma 1.3.13. *For any point $x \in X$, the completed residue field $\widehat{k(x)}$ is a perfectoid field. In particular, there is a morphism $z : \operatorname{Spa}(L, L^+) \to X$ of adic spaces, where L is a perfectoid field, such that x is in the image.*

Remark 1.3.14. This illustrates why it is important to include general integral subrings L^+: The lemma is wrong if we only allow L°, as this does not give all of $|X|$!

We now use Theorem 1.2.37 and untilt z. This yieds a commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}(L^\sharp, L^{\sharp+}) & \xrightarrow{z^\sharp} & X \\ \downarrow{\scriptstyle -^\flat} & & \downarrow{\scriptstyle -^\flat} \\ \mathrm{Spa}(L, L^+) & \xrightarrow{z} & X^\flat. \end{array}$$

It is easy to see that the left map is bijective. Hence x is in the image of the right map.

Step 5: By approximating elements of R by images of $\sharp : R^\flat \to R$, one shows that any rational subspace $U \subseteq X$ is of the form $U'^\sharp$ for some rational subspace $U' \subseteq X^\flat$. By a topological argument for spectral spaces, this shows that $|X| \to |X^\flat|$ is injective. As it is surjective by step 4, another spectral space argument deduces that it is a homeomorphism. In particular, the rational subspace U' is uniquely determined. This shows the remaining statement that $-^\flat$ identifies rational subspaces. ■

By combining Theorem 1.3.11 and Theorem 1.2.37, we deduce via sheafification:

Corollary 1.3.15. *Let X be an affinoid perfectoid space. Then via the identification $|X| \cong |X^\flat|$ we can regard sheaves on $X^\flat$ as sheaves on X. Via this identification, we have an isomorphism*

$$\sharp : \mathcal{O}^+_{X^\flat}/\varpi^\flat \xrightarrow{\sim} \mathcal{O}^+_X/\varpi.$$

Exercise 1.3.16. Prove this. Also show that the sharp maps for rational subspaces can be assembled to a multiplicative morphism of sheaves $\sharp : \mathcal{O}_{X^\flat} \to \mathcal{O}_X$.

Summarising the results of this subsection, the tilting equivalence gives us a very close relation between the affinoid adic spaces defined by R and the adic spaces defined by its tilt $R^\flat$. The proof of sheafiness is quite hard, but “perfectoid in nature” in that it used the tilting equivalence to go back and forth between characteristics.

Remark 1.3.17. Given an affinoid adic space X without Noetherian assumptions, it is usually difficult to verify directly that it is sheafy. The Theorem of Mihara / Buzzard–Verberkmoes that “stably uniform affinoids are sheafy” gives a great criterion. But even proving that X is stably uniform can be tricky in practice, as it can be quite difficult to describe rational localisations explicitly enough to verify technical properties. Perfectoid spaces can actually help with this by giving a new criterion for sheafiness that does not require computation of rational opens: Hansen–Kedlaya have introduced the notion of **sousperfectoid** spaces [HK][SW20, §6.3]. An adic space is sousperfectoid if it can be covered by affinoid opens $\mathrm{Spa}(A, A^+)$ where A admits an injection $A \hookrightarrow A_\infty$ into a perfectoid algebra A_∞ that has a continuous A-module splitting. They show that any sousperfectoid space is stably uniform, hence sheafy.

1.3.1.2 Globalisation. Now that we know that affinoid perfectoid spaces have good localisation properties, we can proceed to the central definition of this lecture course:

Definition 1.3.18. An adic space X over K is called a **perfectoid space** if it has an open cover $X = \cup U_i$ by affinoid perfectoid spaces U_i over K.

It is immediate from Theorem 1.3.11, specifically the compatibility of tilting with the formation of rational subspaces, that the tilting functor $X \mapsto X^\flat$ glues:

Theorem 1.3.19. (1) *There is a natural equivalence of categories*

$$\{\textit{perfectoid spaces over } K\} \to \{\textit{perfectoid spaces over } K^\flat\}, \quad X \mapsto X^\flat.$$

(2) *Let X be a perfectoid space. Then there is a canonical homeomorphism*

$$-^\flat : |X| \xrightarrow{\sim} |X^\flat|$$

that identifies the affinoid perfectoid open subspaces on both sides. With respect to this homeomorphism, we have an isomorphism $\sharp : \mathcal{O}^+_{X^\flat}/\varpi^\flat \xrightarrow{\sim} \mathcal{O}^+_X/\varpi$.

Exercise 1.3.20. Check that this follows from Theorem 1.3.11 and Corollary 1.3.15.

Exercise 1.3.21. Verify that via the identification $|X| = |X^\flat|$ from Theorem 1.3.19, there are also sheaf versions of $\sharp$ and θ on X, namely natural morphisms

$$\sharp : \mathcal{O}_{X^\flat} \to \mathcal{O}_X, \quad \mathcal{O}_{X^\flat} \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} \mathcal{O}_X, \quad \theta : W(\mathcal{O}^+_{X^\flat}) \to \mathcal{O}^+_X.$$

1.3.1.3 Almost purity for perfectoid spaces. There is also a global version of the Almost Purity Theorem from §1.2.2.3. For this we first recall that there is a good notion of étale morphisms of adic spaces:

Definition 1.3.22 ([Sch12, Definition 7.1]). (1) A morphism $f : X \to Y$ of adic spaces over K is **finite étale** if for any affinoid open subspace $U = \mathrm{Spa}(A, A^+) \subseteq Y$, the pullback $f^{-1}(U) = \mathrm{Spa}(B, B^+)$ is affinoid, the induced map $A \to B$ is finite étale and B^+ is the integral closure of A^+ inside B. A morphism $X \to Y$ of adic spaces over K is **étale** if locally on source and target, it is the composition of an open immersion $X \to Z$ followed by a finite étale map $Z \to Y$.

(2) Let X be an adic space[6]. The **étale site** $X_{\text{ét}}$ is the category of étale morphisms $Y \to X$ endowed with the following Grothendieck topology: A collection of morphisms $(f_i : Y_i \to Y)_{i \in I}$ over X is a cover if $\cup_{i \in I} f_i(|Y_i|) = |Y|$.

Lemma 1.3.23. *Let $f : Y \to X$ be an étale morphism of adic spaces over K. If X is a perfectoid space, then so is Y.*

[6]To ensure that $X_{\text{ét}}$ is a site, it is best to impose some technical condition on X that implies sheafiness of étale covers. For example, sousperfectoidness (Remark 1.3.17) is such a property.

Proof. The statement is local, so we may assume that Y is affinoid and X is affinoid perfectoid. By the definition of étale morphisms, f is locally a composition of rational opens and finite étale maps, so we may reduce to these cases. Any open subspace of a perfectoid space is perfectoid by Theorem 1.3.3 (see also Example 1.3.26 below). If $f : Y \to X$ is finite étale, then $\mathcal{O}(X) \to \mathcal{O}(Y)$ is finite étale, and it follows from Corollary 1.2.43 that Y is affinoid perfectoid. ∎

Theorem 1.3.24. *For any perfectoid space X, tilting induces an equivalence of sites*

$$-^\flat : X_{\text{ét}} \xrightarrow{\sim} X^\flat_{\text{ét}}$$

that identifies the finite étale objects on both sides.

Exercise 1.3.25. Check that this follows from Theorem 1.3.19 and Lemma 1.3.23.

1.3.1.4 Examples of perfectoid spaces.

Example 1.3.26. Any open subspace of a perfectoid space is perfectoid: This follows from Theorem 1.3.11 as rational opens form a basis of $|X|$ for any affinoid X.

Exercise 1.3.27. An example of an affinoid perfectoid space is the perfectoid ball

$$\mathbb{B}_\infty := \operatorname{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$$

associated to the perfectoid algebra of Example 1.2.7 and Proposition 1.2.8. Show that this represents the functor that sends any perfectoid space Y over K to $H^0(Y, \mathcal{O}^+_{Y^\flat})$.

Example 1.3.28. An example of a perfectoid subspace that is not affinoid perfectoid: Let $X = \mathbb{B}_\infty$ be the perfectoid unit ball from Exercise 1.3.27. Then the open ball defined by

$$X = \bigcup_{n\in\mathbb{N}} X(|T| \le 1 - \tfrac{1}{p^n})$$

is an example for a non-quasi-compact, hence non-affinoid perfectoid space.

Example 1.3.29 ([Sch15, Definition III.2.18]). Assume that $\operatorname{char}(K) = p$. Let $Y = \operatorname{Spa}(R, R^+)$ be any sheafy affinoid adic space over K, for example a rigid space. Recall from Proposition 1.2.12 that there exists a perfection $(R^{\text{perf}}, R^{+,\text{perf}})$ of (R, R^+). This induces an affinoid perfectoid space $Y^{\text{perf}} := \operatorname{Spa}(R^{\text{perf}}, R^{+,\text{perf}}) \to Y$. It is easy to see that this is a homeomorphism on the underlying sets, and that $-^{\text{perf}}$ commutes with rational localisation. Consequently, due to the sheafiness assumption, we can glue this to a functor which sends any sheafy adic space over K to a perfectoid space

$$Y^{\text{perf}} \to Y$$

such that $|Y^{\mathrm{perf}}| = |Y|$. The map $Y^{\mathrm{perf}} \to Y$ is called the **inverse perfection** of Y, because one can make precise the idea[7] that it is the "projective limit $\varprojlim_F Y$".

Exercise 1.3.30. Prove that $-^{\mathrm{perf}}$ is the right adjoint to the forgetful functor from perfectoid spaces over K to sheafy adic spaces over K.

Example 1.3.31. Let $\mathbb{T} = \mathrm{Spa}(K\langle T_1^{\pm}, \dots, T_d^{\pm}\rangle)$ be the affine torus of dimension d over K. This is a rigid group, with group structure coming from the multiplicative structure. Consider the p-multiplication map $[p] : \mathbb{T} \to \mathbb{T}$, this is given on coordinates by $T_i \mapsto T_i^p$. In particular, in characteristic p this is the relative Frobenius map, and in characteristic 0 it is a "lift of Frobenius" ($\mathbb{T}$ is the adic generic fibre of an admissible formal scheme over O_K, and ϕ has a formal model which reduces to the Frobenius mod p). By Exercise 1.2.11, we get in the limit an affinoid perfectoid space

$$\mathbb{T}_\infty = \mathrm{Spa}(K\langle T_1^{\pm\frac{1}{p^\infty}}, \dots, T_d^{\pm\frac{1}{p^\infty}}\rangle).$$

Example 1.3.32. Let $\mathbb{P}_K^n$ be the rigid analytic projective n-space over K. Consider the morphism

$$\phi : \mathbb{P}_K^n \to \mathbb{P}_K^n, \quad (x_0, \dots, x_n) \mapsto (x_0^p, \dots, x_n^p).$$

If $\mathrm{Char}(K) = p$, this is the relative Frobenius map, so we can form the inverse perfection (Example 1.3.29), i.e. the "inverse limit" of the tower $\dots \xrightarrow{\phi} \mathbb{P}_K^n \xrightarrow{\phi} \mathbb{P}_K^n \xrightarrow{\phi} \mathbb{P}_K^n$ to get a perfectoid space $\mathbb{P}_K^{n,\mathrm{perf}}$. There is a similar construction when $\mathrm{Char}(K) = 0$: Here, ϕ is again a "lift of Frobenius", so in the limit, we get at a perfectoid space $\mathbb{P}_K^{n,\mathrm{perf}}$.

Exercise 1.3.33. Verify this and prove that $\mathbb{P}_K^{n,\mathrm{perf},\flat} = \mathbb{P}_{K^\flat}^{n,\mathrm{perf}}$.

Remark 1.3.34. In particular, by Theorem 1.3.11 we have $|\mathbb{P}_K^{n,\mathrm{perf}}| = |\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}|$. It is very tempting to try to use this to tilt projective varieties! Namely, one could take any closed $Z \subseteq \mathbb{P}^n$, pull this back to $\mathbb{P}_K^{n,\mathrm{perf}}$, tilt, and hope that one can "unperfect" the result to a variety in characteristic p. This usually does not work though: The resulting subspace of $\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}$ is in general transcendental and does not descend to a closed subspace of $\mathbb{P}_{K^\flat}^n$. However, in some cases, it does work! This is used by Scholze in [Sch12] to prove the weight monodromy conjecture for toric varieties.

Example 1.3.35. Let A be an abelian variety of good reduction over K. Let $\mathfrak{A}$ be its formal model over $\mathrm{Spf}(O_K)$. Consider the tower $\dots \to \mathfrak{A} \xrightarrow{[p]} \mathfrak{A} \xrightarrow{[p]} \mathfrak{A}$ of multiplication by p maps. On the fibre over $\mathrm{Spf}(O_K/p)$, the morphism $[p]$ factors through the relative Frobenius morphism. It follows that the adic generic fibre of the limit

[7]It is not literally the limit in the category of adic spaces, but for example one can use the weaker notion of tilde-limits from [Hub96, §2.4][SW13, §2.4].

$\varprojlim_{[p]} \mathfrak{A}$ is a perfectoid space A_∞. We will study this example further in §1.4, see Example 1.4.24.

Example 1.3.36. A famous example with arithmetic applications is the tower of modular curves $(\mathcal{X}(p^n))_{n\in\mathbb{N}}$ with $\Gamma(p^n)$-level structure. Scholze proves in [Sch15] that this becomes perfectoid in the limit $n \to \infty$, leading to the perfectoid modular curve $\mathcal{X}(p^\infty)$

Remark 1.3.37. There is a clear pattern here: In all of the above examples, perfectoid spaces arise as covers of rigid spaces that are "infinitely wildly ramified mod p" in a precise sense, exactly like perfectoid fields arose in Lecture 1 as infinitely wildly ramified towers of local fields. Indeed, this is one way that perfectoid spaces are used in practice: Say we wish to prove a statement about a rigid space Y. Roughly speaking, one strategy is to "pull back" the problem to a perfectoid covering space $Y_\infty \to Y$, which might first look like a much more difficult object, but has favourable technical properties (of which we have already seen a few, notably Almost Purity, and we are going to see more soon). One then tries to solve the problem on Y_∞, and descend the solution back down to Y.

Remark 1.3.38. Warning: It is still an open question whether a perfectoid space that is affinoid as an adic space is affinoid perfectoid. This might first seem like a tautology, but it is not! Namely, it's not clear that a Huber pair (R, R^+) for which $\mathrm{Spa}(R, R^+)$ can be covered by affinoid perfectoids is itself affinoid perfectoid.

1.3.1.5 Fibre products of perfectoid spaces. In general, fibre products of adic spaces can be a bit of a thorny issue, as in general they do not always exist due to issues with sheafiness. Luckily, this is no problem for perfectoid spaces:

Proposition 1.3.39. *Let $X \to Y$, $Z \to Y$ be morphisms of perfectoid spaces over K. Then the fibre product $X \times_Y Z$ exists in the category of adic spaces over K and is a perfectoid space over K. If X, Y, Z are affinoid perfectoid, then so it $X \times_Y Z$.*

Proof. As usually, one reduces to the case that $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$ and $Z = \mathrm{Spa}(C, C^+)$ are affinoid perfectoid. Recall that the images of A^+, B^+ and C^+ in the almost category are perfectoid $\mathcal{O}_K^a$-algebras. One can use this to see:

Exercise 1.3.40. Verify directly that $D_0 := A^+ \hat{\otimes}_{C^+} B^+$ is also a perfectoid $\mathcal{O}_K^a$-algebra

Let now $D := A^+ \hat{\otimes}_{C^+} B^+[\frac{1}{\varpi}]$ and let D^+ be the integral closure of D_0 in D. Then $W := \mathrm{Spa}(D, D^+)$ is affinoid perfectoid. One easily verifies that W is the fibre product. ∎

1.3.2 Almost Acyclicity

Scholze's proof that affinoid perfectoid spaces are sheafy is different to the one above[8]. Instead, he proved the stronger Almost Acyclicity Theorem, which we discuss next.

1.3.2.1 Acyclicity of $\mathcal{O}_X$. Let X be a sheafy affinoid adic space. Then the sheafy hypothesis automatically guarantees that $\mathcal{O}_X$ is not only a sheaf but in fact an **acyclic** sheaf. This means that

$$H^n(X, \mathcal{O}_X) = 0 \quad \text{for } n \geq 1.$$

Equivalently, for any affinoid cover $\mathfrak{U}$ of X, the augmented Čech complex

$$\check{C}^*(\mathfrak{U}, \mathcal{O}_X) = \left[\mathcal{O}_X(X) \to \prod_{U \in \mathfrak{U}} \mathcal{O}_X(U) \to \prod_{U_{1,2} \in \mathfrak{U}} \mathcal{O}_X(U_1 \cap U_2) \to \dots\right]$$

is automatically exact if it is left-exact. Recall now that the integral subsheaf $\mathcal{O}_X^+$ is also a sheaf – this is automatic from the description that for $f \in \mathcal{O}_X(U)$, we have

$$f \in \mathcal{O}_X^+(U) \Leftrightarrow |f(x)| \leq 1 \text{ for all } x \in U.$$

However, it is not true in general that $H^n(X, \mathcal{O}_X^+)$ vanishes for any $n \geq 1$. In fact, this already fails for many rigid spaces. Since $\mathcal{O}_X = \mathcal{O}_X^+[\frac{1}{\varpi}]$ and formation of cohomology commutes with inverting ϖ (because X is quasi-compact), we know that

$$H^n(X, \mathcal{O}_X^+)[\tfrac{1}{\varpi}] = H^n(X, \mathcal{O}_X) = 0 \quad \text{for } n \geq 1. \tag{1.4}$$

This shows that any element in $H^n(X, \mathcal{O}_X^+)$ is ϖ-power torsion.

1.3.2.2 Almost Acyclicity of $\mathcal{O}_X^+$. One of the key technical properties of affinoid perfectoid spaces is that the cohomology of $\mathcal{O}^+$ can be controlled for them:

Theorem 1.3.41. *Let X be an affinoid perfectoid space over K. Then for any $n \geq 1$,*

$$H^n(X, \mathcal{O}_X^+) \overset{a}{=} 0.$$

Exercise 1.3.42. Deduce that $H^0(X, \mathcal{O}_X^+)/\varpi \overset{a}{=} H^0(X, \mathcal{O}_X^+/\varpi)$.

Proof. We give a sketch of the proof of Theorem 1.3.41:

[8] In fact, the stably uniform criterion wasn't found yet at the time of his thesis. And it may well not have been found without it. The proof does not use perfectoid spaces, but many people only started studying non-Noetherian adic spaces (or indeed any kind of adic spaces) because of Scholze's work.

Step 1: Once again, one first proves the case of $\mathrm{Char}(K) = p$ and deduces the case of $\mathrm{Char}(K) = 0$ via tilting. We first explain how this reduction to characteristic p works:

Exercise 1.3.43. Suppose we know already that Theorem 1.3.41 holds in characteristic p. Let X be an affinoid perfectoid K-algebra of characteristic 0. Show that we have $H^n(X, \mathcal{O}^+/\varpi) \overset{a}{=} 0$. Hint: Use Corollary 1.3.15.

As usual, we can compute $\mathcal{O}^+$-cohomology by considering Čech cohomology on a basis of the topology, for which we take the rational opens of X. Let thus $\mathfrak{U}$ be any rational open cover of X and consider the Čech complex $\check{C}^*(\mathfrak{U}, \mathcal{O}_X^+)$. By Exercise 1.3.43, we have $H^n(U, \mathcal{O}_X^+/\varpi) \overset{a}{=} 0$. By the Čech-to-sheaf spectral sequence, it follows that $\check{C}^*(\mathfrak{U}, \mathcal{O}_X^+)/\varpi$ is almost exact. As $\check{C}^*(\mathfrak{U}, \mathcal{O}_X^+)$ is a complex of flat ϖ-adically complete $\mathcal{O}_K$-modules, the case of $\mathrm{Char}(K) = 0$ now follows from the following fact:

Exercise 1.3.44. Let $C^\bullet$ be a bounded complex of flat ϖ-adically complete $\mathcal{O}_K$-modules. Suppose that $H^n(C^\bullet/\varpi) \overset{a}{=} 0$ for all n. Then $H^n(C^\bullet) \overset{a}{=} 0$ for all n.

Step 2: We have thus reduced to the case that $\mathrm{Char}(K) = p$. Write $X = \mathrm{Spa}(A, A^+)$. We now further reduce to the case that $X = Y^{\mathrm{perf}}$ is the inverse perfection of a rigid space Y (Example 1.3.29). The idea for this is to write $A^+ = \varinjlim_{i \in I} B_i$ as a colimit of subalgebras B_i of topologically finite type over $\mathcal{O}_K$: This is always possible as $A^+ \subseteq A^\circ$.

Given such a "rigid approximation" of X, write $Y_i := \mathrm{Spa}(B_i[\frac{1}{\varpi}], B_i^+)$ where B_i^+ is the integral closure of B_i in $B_i[\frac{1}{\varpi}]$. Then one can show that for any $n \geq 0$, one has

$$H^n(X, \mathcal{O}^+/\varpi) \overset{a}{=} \varinjlim_{i \in I} H^n(Y_i^{\mathrm{perf}}, \mathcal{O}^+/\varpi).$$

It thus suffices to consider Y_i^{perf}.

Remark 1.3.45. This was a topological algebra analogue of "Noetherian approximation", a standard procedure to reduce algebraic problems to finite type K-algebras.

Step 3: Assume that $X = Y^{\mathrm{perf}}$ for a rigid space Y. Recall from Example 1.3.29 that the natural map $Y^{\mathrm{perf}} \to Y$ induces a homeomorphism $|Y^{\mathrm{perf}}| = |Y|$ which we can use to regard sheaves on Y^{perf} as sheaves on Y. Let us denote the integral structure sheaf of Y by $\mathcal{O}_Y^+$ and the one of Y^{perf} by $\mathcal{O}_{Y^{\mathrm{perf}}}^+$. As before, the goal is to compute $H^n(Y, \mathcal{O}_{Y^{\mathrm{perf}}}^+)$ using Čech cohomology. For this, we start with $H^n(Y, \mathcal{O}_Y^+)$: Let $\mathfrak{U}$ be any rational cover of Y. Recall that the Čech cohomology group

$$\check{H}^n(\mathfrak{U}, \mathcal{O}_Y^+)$$

is by definition the cohomology in degree n of the complex of Čech cochains

$$\prod_{U \in \mathfrak{U}} \mathcal{O}_Y^+(U) \to \prod_{U_{1,2} \in \mathfrak{U}} \mathcal{O}_Y^+(U_1 \cap U_2) \to \dots.$$

Recall that intersections $U_1 \cap \cdots \cap U_k$ of rational open subspaces are again rational, hence affinoid. Therefore, by the same argument as in (1.4), this complex becomes exact in degree ≥ 1 after inverting ϖ due to acyclicity of $\mathcal{O}_X$ on affinoid subspaces.

We now crucially use that each $\mathcal{O}_Y(U_1 \cap \cdots \cap U_k)$ is in particular a K-Banach space. A neat topological algebra argument using Banach's Open Mapping Theorem (see [Bha17, Proposition 9.3.3]) shows that $\check{H}^n(\mathfrak{U}, \mathcal{O}_Y^+)$ is in fact of bounded torsion: There is $0 \neq x \in \mathcal{O}_K$ such that multiplication by x is $= 0$ on $\check{H}^n(\mathfrak{U}, \mathcal{O}_Y^+)$ for any $n \geq 0$.

To compute $\check{H}^n(\mathfrak{U}, \mathcal{O}^+_{Y_{\mathrm{perf}}})$, we now take the limit over F on Y. To distinguish the different copies of Y in the Frobenius tower, let us define $Y_m := Y$ for every $m \in \mathbb{N}$ so that the tower becomes

$$\dots \xrightarrow{F} Y_3 \xrightarrow{F} Y_2 \xrightarrow{F} Y_1 = Y.$$

Now comes the cool trick: Consider the tower of Frobenius on the resulting Čech complexes. This is a tower (written vertically) of complexes (written horizontally)

$$\begin{array}{ccccccc}
\prod_U \mathcal{O}^+_{Y_1}(U) & \longrightarrow & \prod_{U_{ij}} \mathcal{O}^+_{Y_1}(U_i \cap U_j) & \longrightarrow & \prod_{U_{i,j,k}} \mathcal{O}^+_{Y_1}(U_i \cap U_j \cap U_k) & \longrightarrow & \dots \\
\downarrow F & & \downarrow F & & \downarrow F & & \\
\prod_U \mathcal{O}^+_{Y_2}(U) & \longrightarrow & \prod_{U_{ij}} \mathcal{O}^+_{Y_2}(U_i \cap U_j) & \longrightarrow & \prod_{U_{i,j,k}} \mathcal{O}^+_{Y_2}(U_i \cap U_j \cap U_k) & \longrightarrow & \dots \\
\downarrow F & & \downarrow F & & \downarrow F & & \\
\dots & & \dots & & \dots & &
\end{array}$$

where each F is given by $x \mapsto x^p$. Note that these maps are not K-linear, rather they are semilinear with respect to $F : K \to K$, $x \mapsto x^p$. Hence the morphism of sheaves

$$F : \mathcal{O}_{Y_1} \to \mathcal{O}_{Y_2}$$

becomes K-linear if we instead endow $\mathcal{O}_{Y_2}$ with the K-linear structure via $F : K \to K$. Iterating this construction, to make the diagram K-linear, we need to endow the sections of $\mathcal{O}_{Y_m}$ with a new K-vector space structure $*_m$ defined for any $a \in K$ and any section y of $\mathcal{O}^+_{Y_m}$ by

$$a *_m y = a^{p^m} \cdot y.$$

Having set this all up, the point is now that since $\check{H}^n(\mathfrak{U}, \mathcal{O}_Y^+)$ is killed by x, this means for the new K-linear structure on $\mathcal{O}^+_{Y_m}$ that for any $y \in \check{H}^n(\mathfrak{U}, \mathcal{O}^+_{Y_m})$,

$$x^{1/p^m} *_m y = x \cdot y = 0.$$

Hence $\check{H}^n(\mathfrak{U}, \mathcal{O}^+_{Y_m})$ is killed by x^{1/p^m}. Since $v(x^{1/p^m}) \to 0$ for $m \to \infty$, we have $\mathfrak{m}_K = (x^{1/p^m}, m \in \mathbb{N})$. So

$$\varinjlim_F \check{H}^n(\mathfrak{U}, \mathcal{O}^+_{Y_m}) \stackrel{a}{=} 0$$

with respect to its K-linear structure via $*_m$ in the m-th entry.

It is then easy to see using Exercise 1.3.44 applied to the augmented Čech complex that the same still holds after completing the complex ϖ-adically. Consequently,

$$\check{H}^n(\mathfrak{U}, \mathcal{O}^+_{Y^{\mathrm{perf}}}) \overset{a}{=} 0,$$

as we wanted to see. ■

Remark 1.3.46. The same argument in cohomological degree 0 shows that

$$\check{H}^0(\mathfrak{U}, \mathcal{O}^+_{Y^{\mathrm{perf}}}) \overset{a}{=} \mathcal{O}^+_Y(Y)^{\mathrm{perf}}.$$

After inverting p it follows for $X = Y^{\mathrm{perf}}$ that $\check{H}^0(\mathfrak{U}, \mathcal{O}_X) = \mathcal{O}_X(X)$. This is Scholze's proof that $\mathcal{O}_X$ (and hence automatically $\mathcal{O}_X^+$) is a sheaf on perfectoid spaces: It basically arises as a side product of the proof of the Almost Acyclicity Theorem.

1.3.2.3 A variant for the étale topology. The same line of argument also works in the étale topology, i.e. for the site $X_{\text{ét}}$ from Definition 1.3.22. For this one additionally uses the Tilting Equivalence for étale sites Theorem 1.3.24:

Theorem 1.3.47. *Let X be an affinoid perfectoid space over K. Then $\mathcal{O}_X$ is a sheaf on $X_{\text{ét}}$ and*

$$H^n_{\text{ét}}(X, \mathcal{O}_X^+) := H^n(X_{\text{ét}}, \mathcal{O}_X^+) \overset{a}{=} 0 \quad \textit{for } n \geq 1.$$

Remark 1.3.48. This result is even more remarkable than the version of Almost Acyclicity for the analytic topology: Let X be instead an affinoid rigid space. While it is still possible in good cases to control $H^*_{\mathrm{an}}(X, \mathcal{O}^+)$, the étale cohomology groups $H^*_{\text{ét}}(X, \mathcal{O}^+)$ tend to be difficult to control. For example, even for the unit disc $\mathbb{B}^1$ over an algebraically closed field (one of the nicest and most well-behaved adic spaces that you could possibly imagine), we have $H^i_{\mathrm{an}}(\mathbb{B}^1, \mathcal{O}^+) = 0$ for $i \geq 1$, but the étale cohomology group $H^2_{\text{ét}}(\mathbb{B}^1, \mathcal{O}^+)$ is huge and has unbounded p-torsion (see Exercise 1.4.42 below). A related issue is that while $H^n_{\mathrm{an}}(X, \mathcal{O}) = H^n_{\text{ét}}(X, \mathcal{O})$ for any rigid space X, the natural map

$$H^n_{\mathrm{an}}(X, \mathcal{O}^+) \to H^n_{\text{ét}}(X, \mathcal{O}^+)$$

is usually not an isomorphism.

Remark 1.3.49. Note that even though we have already proved that X is a sheafy adic space, it does not immediately follow that $\mathcal{O}_X$ is also a sheaf on $X_{\text{ét}}$. For rigid spaces, this is known by the work of Huber [Hub96] (which is required for the proof).

But in fact, there would also be a more direct way to deduce this from sheafiness of X: Namely, under very mild assumptions, [KL15, Theorem 8.2.22] says that it is more generally true that sheafiness of $\mathcal{O}_X$ on X implies sheafiness of $\mathcal{O}_X$ on $X_{\text{ét}}$.

In conclusion, we can add Almost Acyclicity of $\mathcal{O}^+$ to our list of "favourable properties of perfectoid spaces" and keep in mind that étale cohomology of $\mathcal{O}^+$ is much more manageable for perfectoid spaces than it usually is, even for rigid spaces.

1.4 Applications to p-adic Hodge theory

As an application of the discussion so far, we now give an example of how perfectoid spaces are used in practice, in applications to p-adic Hodge theory. For this we will roughly follow [Sch13a, §3-§4]. As a first step, we need to introduce a topology on a rigid space which is much finer than the étale topology, so it "sees perfectoid spaces":

1.4.1 The pro-étale site

Recall that for any scheme S, we have a hierarchy of topologies

$$S_{\text{fpqc}} \longrightarrow S_{\text{fppf}} \longrightarrow S_{\text{ét}} \longrightarrow S_{\text{Zar}},$$

written as morphisms of sites. For an adic space X over K, we have seen the topologies

$$X_{\text{ét}} \to X_{\text{an}}$$

where X_{an} refers to the site associated to the topological space $|X|$. Already for these sites, proving that $\mathcal{O}_X$ is a sheaf was hard. But it is a priori not clear how to find an analogue of the fpqc topology in the analytic context. As it turns out, the key to finer sites on X is to consider categories over X that are large enough to contain perfectoid spaces, with topologies that are "locally perfectoid". To make this precise, we take inspiration from the examples on perfectoid spaces in §1.3.1.4: In particular, the perfectoid cover

$$\mathbb{T}_\infty := (\to \dots \xrightarrow{[p]} \mathbb{T} \xrightarrow{[p]} \mathbb{T})$$

of the torus from Example 1.3.31, or the perfectoid cover $E_\infty \to E$ of an elliptic curve defined by the tower $\to \dots \xrightarrow{[p]} E \xrightarrow{[p]} E$ from Example 1.3.35. Both of these are examples of towers of finite étale maps over a rigid space that "become perfectoid in the limit". Our aim is to introduce a topology on X that refines the étale topology and includes such towers as covers.

1.4.1.1 Definition of the pro-étale site. From now on, we fix a non-archimedean field K over $\mathbb{Q}_p$, i.e. we assume that K has characteristic 0 and residue characteristic p. Let X be a rigid space over K, considered as an adic space. Roughly following [Sch13a, Definition 3.9], we now define the pro-étale site of X:

Definition 1.4.1. We define a category $X_{\text{proét}}$ as follows:

(1) Objects are the small cofiltered inverse systems $(C_i)_{i\in I}$ of objects of $X_{\text{ét}}$ for which there is $j \in I$ so that for $i \geq j$, the map $X_i \to X_j$ is surjective finite étale.

(2) The morphisms are the morphisms between pro-objects. Explicitly, given $C = (C_i)_{i\in I}$ and $D = (D_j)_{j\in J}$ in $X_{\text{proét}}$, this means that we have

$$\operatorname{Map}_{X_{\text{proét}}}(C, D) = \varprojlim_{j\in J} \varinjlim_{i\in I} \operatorname{Map}_{X_{\text{ét}}}(C_i, D_j).$$

Remark 1.4.2. What does "small" mean in (1): We would like to allow I to be any small index category. Unfortunately, there are genuine set-theoretic issues! This is solved by fixing a cut–off cardinal κ, and only allowing index sets of cardinality $\leq \kappa$, see [Sch16, (1)][Sch18, §4]. In practice, one often uses countable inverse systems, for which this issue does not arise. For our purposes, we will thus ignore this issue.

For example, the pro-systems mentioned just before §1.4.1.1 define objects in $X_{\text{proét}}$. To endow this category with a topology, we need to introduce the following:

Definition 1.4.3. Any object $C = (C_i)_{i\in I}$ in $X_{\text{proét}}$ has an associated topological space

$$|C| := \varprojlim_{i\in I} |C_i|$$

that we can use e.g. to speak of an analytic cover of C. This construction is functorial, i.e. morphisms in $X_{\text{proét}}$ induce continuous maps of the associated topological spaces.

Definition 1.4.4. Let $f : (C_i)_{i\in I} \to D = (D_i)_{i\in I}$ be a morphism in $X_{\text{proét}}$.

(1) f is called **étale** if there is a morphism $C_i \to D_i$ in $X_{\text{ét}}$ for some $i \in I$ such that f is isomorphic to $D \times_{D_i} C_i \to D$.
(2) f is **finite étale** if moreover $C_i \to D_i$ can be taken to be finite étale.
(3) f is **pro-étale** if $f = \varprojlim f_j$ for a cofiltered inverse system of étale maps $(f_j : C^j \to D)_{j\in J}$ with $C^j \in X_{\text{proét}}$ for which $\exists k \in J$ such that $C^j \to C^k$ is finite étale for all $j \geq k$.

Definition 1.4.5. We endow $X_{\text{proét}}$ with a Grothendieck topology as follows: A family of morphisms $(f_j : C^j \to D)_{j\in J}$ in $X_{\text{proét}}$ is a cover if each f_j is pro-étale and

$$\textstyle\bigcup_{j\in J} f_j(|C^j|) = |D|.$$

We call $X_{\text{proét}}$ endowed with this Grothendieck topology the **pro-étale site** of X.

Lemma 1.4.6 ([Sch13a, Lemma 3.10]). *The pro-étale site is a site.*

Given that $X_{\text{proét}}$ is a site, it is clear that there is a natural morphism of sites

$$\nu : X_{\text{proét}} \to X_{\text{ét}}$$

sending any étale morphism $Y \to X$ to the pro-system (Y) indexed over the singleton.

Let us give some less trivial examples of objects in the pro-étale site:

Example 1.4.7. The system $\ldots \xrightarrow{[p]} \mathbb{T} \xrightarrow{[p]} \mathbb{T}$ in Example 1.3.31 is a pro-étale cover of $\mathbb{T}$. The system $\ldots \xrightarrow{[p]} E \xrightarrow{[p]} E$ in Example 1.3.35 is a pro-étale cover of E.

Example 1.4.8. To any finite set S, we can associate an object $\underline{S}$ of $X_{\text{ét}}$, defined as the morphism $\sqcup_S X \to X$. For any profinite set $S = \varprojlim S_i$, we thus obtain an object

$$\underline{S} := \varprojlim_{i\in I} \underline{S_i} = \varprojlim_{i\in I} \sqcup_{S_i} X \in X_{\text{proét}}.$$

This is a pro-finite-étale cover of X. We can use this to explain the relevance of the "pro-étale" assumption in the definition of covers: Namely, we note that $\underline{S}$ is different from the disjoint union $\sqcup_S X$: For example, if $X = \mathrm{Spa}(K)$, then $|\underline{S}|$ is S endowed with its natural profinite topology, whereas $|\sqcup_S X| = S^{\mathrm{disc}}$ is the topological space consisting of S endowed with the discrete topology. There is clearly a natural map $f : \sqcup_S X \to \underline{S}$, which induces the identity $S^{\mathrm{disc}} \to S$ on the underlying topological spaces. This is an example of a morphism in $X_{\mathrm{proét}}$ that is a set-theoretic cover, but is not considered to be a cover in $X_{\mathrm{proét}}$. For example, this follows from the observation that $\underline{S}$ is quasi-compact, but $\sqcup_S X$ admits no finite refinement covering $\underline{S}$.

Example 1.4.9. More generally, let T be a topological space that is the disjoint union[9] $T = \sqcup T_i$ of profinite sets T_i. Then we associate to T the object $\underline{T} := \sqcup \underline{T_i}$ in $X_{\mathrm{proét}}$. Its underlying topological space is $|\underline{T}| = T$. We also denote by $\underline{T}$ the sheaf on $X_{\mathrm{proét}}$ that it represents. We call this the locally profinite sheaf associated to T. For example, $\mathbb{Q}_p$ is a disjoint union of profinite sets, so we can now regard $\underline{\mathbb{Q}_p}$ as a sheaf on $X_{\mathrm{proét}}$.

Example 1.4.10. For $X = \mathrm{Spa}(\mathbb{C}_p)$, any étale cover $Y \to X$ is split, hence of the form $\sqcup_S X \to X$ for some (not necessarily finite) set S. It follows from this that the functor

$$\{\text{(small) disjoint unions of profinite sets}\} \to \mathrm{Spa}(\mathbb{C}_p)_{\mathrm{proét}}$$

from Example 1.4.9 is an equivalence of categories. It becomes an equivalence of sites when we endow the left hand side with the topology in which a family $(f_i : S_i \to T)_{i \in I}$ is a cover if there is a finite subset $J \subseteq I$ such that $T = \cup_{j\in J} f_j(S_i)$ and the f_j are open. In particular, the topos of sheaves on the site $\mathrm{Spa}(\mathbb{C}_p)_{\mathrm{proét}}$ is then equivalent to the topos of sheaves on the category of (small) profinite sets.

This is a good point to mention that Example 1.4.10 is related to Clausen–Scholze's theory of "condensed mathematics" [Sch]:

Remark 1.4.11. There is also an algebraic analogue for the pro-étale site, for schemes instead of adic spaces: Bhatt–Scholze [BS15, §4] have defined a pro-étale site $S_{\mathrm{proét}}$ for any scheme S. Let us mention what this looks like for a geometric point:

Similarly as in Example 1.4.10, for $S = \mathrm{Spec}(\mathbb{C}_p)$, the qcqs objects of $\mathrm{Spec}(\mathbb{C}_p)_{\mathrm{proét}}$ turn out to be identified with profinite sets. But this time, the induced Grothendieck topology on profinite sets is instead defined by allowing any jointly surjective covers (i.e. there is no openness condition), see [BS15, Example 4.1.10]. We refer to [BS15, Remark 4.1.11] for more details on the comparison to $\mathrm{Spa}(\mathbb{C}_p)_{\mathrm{proét}}$.

Condensed sets in the sense of Clausen–Scholze can now be defined as sheaves on $\mathrm{Spec}(\mathbb{C}_p)_{\mathrm{proét}}$. Historically, this was the starting point of condensed mathematics.

[9] Equivalently, T is a Hausdorff space which is locally profinite and has cohomological dimension 0, see [Wie69, Theorem 5.1]. We thank the referee for pointing this out.

1.4.1.2 Étale cohomology via $X_{\text{proét}}$. Before discussing the connection to perfectoid spaces, let us also mention right away a very nice application of the pro-étale site: It can be used to define **étale cohomology with $\mathbb{Q}_p$-coefficients** of a rigid space.

As in algebraic geometry, étale geometry of a rigid space X is usually defined as

$$H^n_{\text{ét}}(X, \mathbb{Q}_p) := \varprojlim_{k\in\mathbb{N}} H^n_{\text{ét}}(X, \mathbb{Z}/p^k\mathbb{Z})[\tfrac{1}{p}].$$

Via $X_{\text{proét}}$, there is a cleaner way to do this, using the sheaf $\underline{\mathbb{Q}}_p$ from Example 1.4.9:

Proposition 1.4.12. *Let X be a qcqs rigid space. Then there is a natural isomorphism*

$$H^n_{\text{ét}}(X, \mathbb{Q}_p) = H^n(X_{\text{proét}}, \underline{\mathbb{Q}}_p).$$

Note that on the left hand side, the cohomology is "just notation", but on the right we are really taking sheaf cohomology on a site!

1.4.1.3 Structure sheaves on $X_{\text{proét}}$. Recall the morphism of sites $\nu : X_{\text{proét}} \to X_{\text{ét}}$. This induces a structure sheaf $\mathcal{O}_{X_{\text{proét}}} = \nu^*\mathcal{O}_{X_{\text{ét}}}$ on $X_{\text{proét}}$ with integral subsheaf $\mathcal{O}^+_{X_{\text{proét}}} := \nu^*\mathcal{O}^+_{X_{\text{ét}}}$. As before, we often drop the index $-_{X_{\text{proét}}}$ when the site is clear from context.

Lemma 1.4.13 ([Sch13a, Lemma 3.16]). *For $C = (C_i)_{i\in I}$ in $X_{\text{proét}}$ with all C_i qcqs,*

$$\mathcal{O}(C) = \varinjlim_{i\in I} \mathcal{O}(C_i), \quad \mathcal{O}^+(C) = \varinjlim_{i\in I} \mathcal{O}^+(C_i).$$

This is not immediate from the definition: A priori, $\nu^*\mathcal{O}$ is the *sheafification* of the presheaf described in the lemma. The lemma says that this presheaf is a sheaf.

Example 1.4.14. For $\mathbb{T}_\infty := (\dots \xrightarrow{[p]} \mathbb{T}^d)$ in $\mathbb{T}^d_{\text{proét}}$, the sheaf $\mathcal{O}^+$ evaluates to

$$\mathcal{O}^+(\mathbb{T}_\infty) = \varinjlim_{n\in\mathbb{N}} \mathcal{O}_K\langle T^{\pm\frac{1}{p^n}}\rangle.$$

This is different from the algebra $\mathcal{O}_K\langle T^{\pm\frac{1}{p^\infty}}\rangle$ of functions on the perfectoid space $\mathbb{T}_\infty$ from Example 1.2.10 and Example 1.3.31.

As this example shows, in order to see perfectoid algebras appearing as algebras of functions on objects of $X_{\text{proét}}$, we need to form the sheaf-theoretic p-adic completion

$$\widehat{\mathcal{O}}^+ := \varprojlim_{k\in\mathbb{N}} \mathcal{O}^+/p^k, \quad \widehat{\mathcal{O}} := \widehat{\mathcal{O}}^+[\tfrac{1}{p}].$$

The sheaf $\widehat{\mathcal{O}}$ on $X_{\text{proét}}$ is a priori difficult to compute explicitly: Already $\mathcal{O}^+/p^k$ shows unexpected behaviour on rigid spaces, since $H^1_{\text{ét}}(X, \mathcal{O}^+)$ may be large (Remark 1.3.48). For example, for any affinoid rigid space X, we have a natural map

$$\mathcal{O}(X) \to \widehat{\mathcal{O}}(X),$$

but despite the fact that $\mathcal{O}^+(X)$ is usually p-adically complete, it's not at all clear that this is an isomorphism. In fact, this can fail in general! For example, it will follow from Remark 1.4.20 below that $\widehat{\mathcal{O}}(X)$ is always reduced, so any nilpotent elements of $\mathcal{O}(X)$ are in the kernel of the above map. That said, we will later see (in Theorem 1.4.36, see also §1.4.2.2):

Proposition 1.4.15. *Assume that X is a smooth rigid space, then $\mathcal{O}(X) = \widehat{\mathcal{O}}(X)$.*

The key to understanding $\widehat{\mathcal{O}}$ are perfectoid objects in $X_{\text{proét}}$, as we now discuss.

1.4.1.4 Perfectoid objects in $X_{\text{proét}}$ and perfectoid covers.

Definition 1.4.16. We call an object $U = (U_i)_{i\in I}$ in $X_{\text{proét}}$ **affinoid perfectoid** if the $U_i = \mathrm{Spa}(R_i, R_i^+)$ are affinoid and $R^+ := (\varinjlim_{i\in I} R_i^+)^\wedge$ is such that $R := R^+[\frac{1}{p}]$ is a perfectoid K-algebra. We call an object **perfectoid** if it is analytic-locally perfectoid (here the analytic topology refers to the topological space defined in Definition 1.4.3).

Lemma 1.4.17 ([Sch13a, Lemma 4.10]). *In the situation of Definition 1.4.16,*

$$\widehat{\mathcal{O}}_X(U) = R, \quad \widehat{\mathcal{O}}_X^+(U) = R^+, \quad \widehat{\mathcal{O}}_X^+/p(U) \overset{a}{=} R^+/p.$$

In particular, sending U to $\mathrm{Spa}(R, R^+)$ glues along rational subspaces, so that we can associate to any perfectoid object Z in $X_{\text{proét}}$ a natural perfectoid space $\widehat{Z} \to X$.

The crucial property of the pro-étale site is now that it is "locally perfectoid": First, a technical lemma ([Sch13a, Lemma 4.5]) using Almost Purity, Theorem 1.2.41, shows that fibre products of étale maps $Z \to Y$ with perfectoid objects $C \to Y$ in $X_{\text{proét}}$ are again perfectoid. By an easy limit argument, one can deduce:

Lemma 1.4.18. *Let Y be any perfectoid object of $X_{\text{proét}}$ and let $Z \to Y$ be a pro-étale morphism in $X_{\text{proét}}$. Then Z is again perfectoid.*

We deduce the following key observation:

Proposition 1.4.19. *If X is smooth, then perfectoid objects form a basis of $X_{\text{proét}}$.*

Proof. By Lemma 1.4.18, it suffices to see that X has a pro-étale cover by a perfectoid space. For this we use that any smooth rigid space X can be covered by open subspaces U that admit an étale morphism $U \to \mathbb{T}^d$ to some torus. We can now form the fibre product inside of $\mathbb{T}^d_{\text{proét}}$

$$\begin{array}{ccc} U_\infty & \longrightarrow & \mathbb{T}^d_\infty \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbb{T}^d \end{array}$$

where $\mathbb{T}^d_\infty \to \mathbb{T}^d$ is the pro-étale perfectoid cover from Example 1.3.31. Then $U_\infty \to \mathbb{T}^d_\infty$ is étale. As $\mathbb{T}^d_\infty$ is perfectoid, Lemma 1.4.18 shows that so is U_∞. ∎

Remark 1.4.20. A more involved argument due to Colmez [Col02] shows that Proposition 1.4.19 is true without smoothness assumption, see [Sch13a, Proposition 4.8].

Remark 1.4.21. The pro-étale site is related to earlier constructions of Faltings, namely the theory of the Faltings topos [Fal02]. From this perspective, Scholze's key new technical ingredients is the theory of perfectoid spaces.

We note that $\mathbb{T}^d_\infty \to \mathbb{T}^d$ is in fact a Galois cover, in the following sense:

Definition 1.4.22. Let G be a profinite group and let $\underline{G}$ be the associated sheaf on $X_{\text{proét}}$ from Example 1.4.8. We say that a morphism $f : V \to U$ in $X_{\text{proét}}$ is a **Galois cover with group** G if f is a pro-étale cover and there is an action[10] $m : \underline{G} \times V \to V$, leaving f invariant and making the following diagram Cartesian:

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle f} \\ \underline{G} \times V & \xrightarrow{\mathrm{pr}_2} & V \end{array}$$

Here the bottom map is the projection to the second factor.

Exercise 1.4.23. Show that $\mathbb{T}^d_\infty \to \mathbb{T}^d$ is a Galois cover with group $\Delta := \mathbb{Z}_p(1)^d$.

We have seen that toric charts are one way to obtain perfectoid pro-étale covers of rigid spaces. In practice, there are often other ways to find perfectoid covers:

Example 1.4.24. Assume that K is algebraically closed and let A be any abelian variety over K of good reduction, considered as an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ by analytification (see §1.10 in [Hüb]). Then $[p^n] : A \to A$ is a finite étale Galois cover with group $A[p^n]$. Let now

$$A_\infty := (\cdots \to A \xrightarrow{[p]} A) \in A_{\text{proét}}.$$

Then it follows from Example 1.3.35 that A_∞ is a perfectoid object of $A_{\text{proét}}$. In fact, by taking the limit of the Cartesian diagram expressing that $[p^n] : A \to A$ is Galois in $A_{\text{proét}}$, we see that $A_\infty \to A$ is a Galois cover for the profinite group T_pA. Here $T_pA = \varprojlim_{n\in\mathbb{N}} A[p^n](K)$ is the Tate module of A. This leads to a p-adic analogue of Riemann uniformization: We have an isomorphism $A = A_\infty/\underline{T_pA}$ of sheaves on $A_{\text{proét}}$.

In fact, all of the above statements still hold for any abelian variety A, not necessarily of good reduction [BGH+22, Theorem 1]. This natural example of a perfectoid pro-étale cover will again play a role in §1.4.3.

[10] This means that it is a morphism in $X_{\text{proét}}$ such that the obvious diagrams in $X_{\text{proét}}$, expressing associativity and the trivial effect of the neutral element, commute.

Example 1.4.25 (Hansen, [BGH+22, Corollary 5.6]). Let K be algebraically closed. Let C be a smooth projective curve over K of genus ≥ 1, regarded as an adic space. Fix a point $x \in C(K)$. Consider the cofiltered inverse system C_∞ of pairs (C', x') consisting of a connected finite étale cover $C' \to C$ together with a fixed lift $x' \in C'(K)$ of x. Then $C_\infty \in C_{\text{proét}}$ is perfectoid, and $C_\infty \to C$ is a Galois cover with group $\pi_1^{\text{ét}}(C, x)$.

1.4.1.5 Pro-étale cohomology. The crucial point for our applications is now that in line with Theorem 1.3.47, affinoid perfectoid objects have good cohomological properties. Namely, we have the following pro-étale variant of Almost Acyclity:

Theorem 1.4.26 (Almost Acyclity). *For any affinoid perfectoid object U in $X_{\text{proét}}$,*

$$H^n_{\text{proét}}(U, \widehat{\mathcal{O}}^+_X) \overset{a}{=} 0 \quad \textit{for all } n \geq 1.$$

Proof sketch: Let $\widehat{U}$ be the affinoid perfectoid space associated to U. One first shows

$$H^n_{\text{proét}}(U, \widehat{\mathcal{O}}^+/p^n) = H^n_{\text{proét}}(U, \mathcal{O}^+/p^n) = H^n_{\text{ét}}(\widehat{U}, \mathcal{O}^+/p^n).$$

By Theorem 1.3.47, this is $\overset{a}{=} 0$. The result then follows by a limit argument. See [Sch13a, Lemma 4.10] for details. ∎

1.4.2 Aside: Other locally perfectoid topologies

1.4.2.1 The v-topology. Over the past years, the pro-étale site has proved to be very useful. But for some applications, it turns out that it is still not fine enough. For example, in this note we have defined $X_{\text{proét}}$ only for rigid spaces. It is straightforward to extend the definition to locally Noetherian adic spaces, but these days (not least because of perfectoid geometry) we often wish to avoid Noetherian hypotheses.

In the meantime, Scholze has introduced in [SW20][Sch18] several finer topologies on analytic adic spaces. The most important one among these is probably the **v-topology**:

Definition 1.4.27. Let X be any adic space over K. The v-site X_v is defined as follows:

(1) The objects are the morphisms $f : T \to X$ from perfectoid spaces T over K.

(2) The morphisms are the morphisms of perfectoid spaces over X.

(3) The covers are the families $(f_i : S_i \to T)_{i \in I}$ such that for any quasi-compact $U \subseteq T$, there is a finite subset $J \subseteq I$ and quasi-compact open subspaces $V_j \subseteq S_j$ for each $j \in J$ such that $|U| = \cup_{j \in J} f_j(|V_j|)$.

If X is a rigid space (or more generally, a locally Noetherian adic space) over $\mathbb{Q}_p$, there is a morphism of topoi

$$X_v^\sim \to X_{\text{proét}}^\sim.$$

The point is that perfectoid objects of $X_{\text{proét}}$ form a basis for the topology by Proposition 1.4.19, so any sheaf on $X_{\text{proét}}$ is determined by its values on perfectoid objects. Thus we might just as well restrict the category of test objects to perfectoid spaces! This is the philosophy that is fully embraced by the v-site.

Assumpton (3) is analogous to the "qc" part of the "fpqc" topology, which is usually the finest topology that appears in algebraic geometry. But in the definition of the v-topology, there is no analogue of the faithfully flat assumption, i.e. of the "fp". In fact, there are no assumptions whatsoever on the morphism f. So it might first seem completely unreasonable that this has good properties. But in fact this works out very well! For example, for any affinoid perfectoid $f : T = \mathrm{Spa}(R, R^+) \to X$,

$$H^n_v(T, \mathcal{O}^+) \overset{a}{=} \begin{cases} R^+ & n = 0, \\ 0 & n > 0. \end{cases}$$

This is a very, very strong form of Almost Acyclicity!

Remark 1.4.28. One can also endow the category of perfectoid spaces over K with a pro-étale topology, leading to a "big pro-étale" site of X whose topos is in between those of $X_{\text{ét}}$ and X_v. Second, in [Sch18] there is also a different kind of small "pro-étale" site which is finer than $X_{\text{proét}}$. So one sometimes has to be a careful which pro-étale site people refer to. But often the difference does not matter in practice.

1.4.2.2 A glimpse on diamonds. Scholze uses the v-topology to define the category of **diamonds**, which extremely vaguely speaking (and simplifying a bit) is a very large category of sheaves on $\mathrm{Spa}(K)_v$ that is small enough that one can "do geometry" with them. They were introduced in [SW20] and developed systematically in [Sch18].

Slightly less vaguely, a diamond is a v-sheaf Y that admits a "pro-étale cover" [11]$X \to Y$ by a perfectoid space X. Since pro-étale covers of perfectoid spaces are perfectoid, every v-sheaf $X \times_Y \cdots \times_Y X$ appearing in the Čech nerve of $X \to Y$ is again represented by a perfectoid space. In practice, one often needs an additional topological condition on Y (a bit similar to a qcqs assumption) which leads to the notion of **locally spatial diamonds**. There is then a functor $X \mapsto X^\diamond$ from analytic adic spaces over K into locally spatial diamonds over K. Any locally spatial diamond has an étale site, which for $X^\diamond$ recovers the étale site of the adic space X. In this sense, diamonds are a huge enlargement of the category of analytic adic spaces over $\mathbb{Z}_p$.

A vague analogy for this diamantine perspective may be that any analytic adic space over $\mathbb{Z}_p$ can be reconstructed[12] from perfectoid charts and gluing data between

[11] It is not immediately clear how to define pro-étale covers of v-sheaves. In fact, the correct technical notion is that of a "'quasi-pro-étale cover".

[12] Precisely how much mathematical structure can be reconstructed depends on how well-behaved the adic space is. For example, perfectoid spaces are always reduced and therefore

them, much in the same way that real manifolds are built out of pieces of $\mathbb{R}^n$ with gluing data between them. In fact, technically speaking, a closer analogy is that diamonds relate to perfectoid spaces like compact Hausdorff spaces relate to profinite sets: Any compact Hausdorff space T admits a surjection by a profinite set $S \to T$ such that every space $S \times_T \cdots \times_T S$ in the Čech nerve is again profinite. Hence T is uniquely determined by the presheaf on the category of profinite sets which it represents. We refer to [Sch18, Example 11.12] for more details on this story.

Following the philosophy suggested by these analogies, perfectoid spaces have thus become the fundamental building blocks of our p-adic geometric world!

1.4.3 p-adic Hodge theory

While perfectoid spaces themselves are a revolutionary new concept, many ideas surrounding them have their roots in the much older field of p-adic Hodge theory due to Tate, Fontaine, Faltings, ... This is where instances of many related constructions first appeared, including perfectoid fields, Fontaine's θ-map, the first instance of the tilting construction, almost mathematics, the predecessor of almost purity...

It is therefore not surprising that one of the areas that perfectoid spaces have revolutionised in the past decade is p-adic Hodge theory: They have been used not only to reorganise known results, but they have moreover lead to many completely new concepts like the pro-étale site, $\mathbb{A}_{\mathrm{inf}}$-cohomology, or prismatic cohomology, which have been used to prove long-standing open problems in the field. Let us mention just a few concrete applications to give some examples:

(1) the extension of the Hodge–Tate and de Rham comparison isomorphisms to smooth proper rigid varieties [Sch13b, §3][BMS18, Theorem 1.7], which had already been suggested by Tate [Tat67, p.180]);

(2) various new results on the cohomology of Shimura varieties due to Scholze [Sch15], Pan [Pan22], Rodríguez Camargo [RC22], ..., leading for example to a proof of the rational Calegari–Emerton conjecture [RC22, Theorem 1.1.2];

(3) the prismatic Dieudonné theory of Anschütz–Le-Bras [ALB23] which classifies p-divisible groups over a very general class of base rings;

(4) the relation of p-adic Hodge theory to K-theory furnished by [BMS19], which has been applied by Antieau–Krause–Nikolaus [AKN24] to compute new K-groups, for example of rings like $\mathbb{Z}/p^n\mathbb{Z}$.

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cannot tell a rigid space from its reduction. By Theorems of Kedlaya–Liu and Scholze, there is a fully faithful functor from seminormal rigid spaces over K into diamonds over K, but for any rigid space that's not seminormal, one can't reconstruct the structure sheaf. Seminormality is a very weak regularity criterion though, for example it's weaker than being normal.

As an introductory and hopefully enlightening example for the way that perfectoid spaces help us understand p-adic Hodge theory, we shall now briefly discuss one of these applications, namely the Hodge–Tate comparison. For this we follow Scholze's articles [Sch13a, §4-§5] and [Sch13b, §3]. Let us mention for further reading that a great (and more detailed) exposition of the results discussed in this section is given in [CBC+19].

We start with some general background on p-adic Hodge theory: Generally speaking, this area is concerned with comparing different p-adic cohomology theories.

1.4.3.1 The complex Hodge decomposition. By way of motivation, let us start in complex geometry, and let X be a complex compact Kähler manifold. Four our purposes, you won't miss out on anything important if you just take X to be a complex torus, or in fact you may simply take X to be an elliptic curve over $\mathbb{C}$.

Complex Hodge theory says that there is then a canonical decomposition

$$H^n_{\mathrm{sing}}(X, \mathbb{C}) = \bigoplus_{i+j=n} H^i(X, \Omega^j_X) \tag{1.5}$$

comparing singular cohomology (left hand side) to Hodge cohomology (right hand side). Here Ω^1_X is the vector bundle of holomorphic differentials and $\Omega^j_X := \wedge^j \Omega^1_X$.

More algebraically, if Y is a smooth projective variety over $\mathbb{C}$, then the analytification $X = Y(\mathbb{C})$ is a compact Kähler manifold. The sheaf Ω^j_X is then the analytification of the algebraic vector bundle of Kähler differentials Ω^j_Y. The left hand side of (1.5) is then related to étale cohomology via Artin's Comparison Theorem, which identifies singular cohomology with étale cohomology for torsion coefficients.

1.4.3.2 The Hodge–Tate spectral sequence. Moving on to the p-adic situation, let us for simplicity[13] work over the perfectoid field $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. There is then an analogue of the Hodge decomposition (1.5) over $\mathbb{C}_p$:

In order to formulate this, we first note that the analogue of a compact complex manifold is a proper smooth rigid space X over $\mathbb{C}_p$. For example, you can take X to be the adic analytification of a proper smooth algebraic variety over $\mathbb{C}_p$. In fact, if you wish, you could simply take the rigid analytification of an elliptic curve, for which the discussion will already be meaningful and interesting.

Motivated by the Artin Comparison Theorem and (1.5), the analogue of the left hand side of the Hodge decomposition in p-adic geometry should be given by the étale cohomology[14]

$$H^*_{\text{ét}}(X, \mathbb{C}_p) := H^*_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p.$$

[13] The crucial point being that $\mathbb{C}_p$ is algebraically closed. More generally, we could replace $\mathbb{C}_p$ by any complete algebraically closed non-archimedean field extension C of $\mathbb{Q}_p$.

[14] The right hand side is the correct definition if you work with more general complete algebraically closed fields C.

Remark 1.4.29. Recall that we defined $H^*_{\text{ét}}(X, \mathbb{Q}_p)$ in §1.4.1.5. In line with Proposition 1.4.12, we could equivalently define $H^*_{\text{ét}}(X, \mathbb{C}_p)$ as $H^*_{\text{proét}}(X, \underline{\mathbb{C}}_p)$ where $\underline{\mathbb{C}}_p$ is the pro-étale sheaf from Example 1.4.9, justifying the notation on the left hand side.

On the other side of the Hodge decomposition, Hodge cohomology can be defined for X in exactly the same way as in algebraic geometry, using the sheaf of continuous Kähler differentials Ω^j_X that one can define for any adic space over K.

We can now compare these two cohomology theories as follows:

Theorem 1.4.30 (Hodge–Tate comparison). *Let X be a smooth proper rigid space over $\mathbb{C}_p$. Then there is a first quadrant spectral sequence*

$$E_2^{ij} = H^i(X, \Omega^j_X(-j)) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{C}_p), \tag{1.6}$$

which is functorial in X. It degenerates at the E_2-page.

Remark 1.4.31. The $(-j)$ is a Tate twist, defined by tensoring with the $\mathbb{Z}_p$-module

$$\mathbb{Z}_p(-j) := \mathrm{Hom}_{\mathbb{Z}_p}(T_p\mu_{p^\infty}(\mathbb{C}_p), \mathbb{Z}_p)^{\otimes j}$$

which is finite free of rank 1. You can safely ignore it if you have not encountered this before: One can always fix a system of p-power unit roots $(\zeta_{p^n})_{n\in\mathbb{N}}$ to trivialise any Tate twists, i.e. choose an isomorphism $\Omega^j_X(-j) \cong \Omega^j_X$ and thus forget about the $(-j)$ in the following. The reason it is usually included is that it is important to keep track of it in the case that X admits a model over a local field endowing both sides with a natural Galois action. In this case, the $(-j)$ tells you that the Galois action needs to be twisted. In particular, this is relevant for explaning the naturality of the Hodge–Tate spectral sequence in X (i.e. to explain the word "functorial").

Remark 1.4.32. Theorem 1.4.30 has a long history with contributions by many mathematicians: It was first proved by Tate for abelian varieties with good reduction over p-adic fields in [Tat67], and conjectured in general. This is also the reason why it was given the name "Hodge–Tate". Faltings [Fal88, III Theorem 4.1][Fal02] and Tsuji [Tsu99] proved it in the algebraic case (namely when X is the analytification of the base-change to $\mathbb{C}_p$ of a smooth proper variety defined over a finite extension of $\mathbb{Q}_p$).

In [Sch13a, Sch13b], Scholze gave a new approach which settles the rigid case. The degeneration at E_2 is in general a difficult additional statement, and is the most recent part of the Theorem: It was proved in general in [BMS18, Theorem 13.3.(ii)].

Remark 1.4.33. If you're not a fan of spectral sequences, there are two ways in which you can avoid them in this context while still getting an interesting statement:

(1) First, you could focus on the part where $i + j = 1$, in which case saying the magic words "5-term exact sequence of low degrees" extracts from The-

orem 1.4.30 a short exact sequence of $\mathbb{C}_p$-vector spaces

$$0 \to H^1(X, \mathcal{O}) \to H^1_{\text{ét}}(X, \mathbb{C}_p) \to H^0(X, \Omega^1_X(-1)) \to 0. \tag{1.7}$$

The "degeneration" part of the statement tells you that this is a short exact. This illustrates the difference between a decomposition and a degenerated spectral sequence: In (1.5), we get in cohomological degree 1 a decomposition

$$H^1_{\text{sing}}(X, \mathbb{C}) = H^1(X, \mathcal{O}) \oplus H^0(X, \Omega^1_X).$$

In the p-adic world, we could choose a splitting of the exact sequence to also get such a decomposition, but then the decomposition is in general not canonical! In particular, it is in general difficult to say in which way it's functorial. The short exact sequence (1.7) is already very interesting. For example, by Kiehl's Theorem, the two outer terms are finite dimensional $\mathbb{C}_p$-vector spaces. Thus, for a short exact sequence like (1.7) to exist, it is necessary that $H^1_{\text{ét}}(X, \mathbb{C}_p)$ is finite dimensional. But a priori this is not at all obvious![15]

(2) Second, there is the following variant: If X comes via base-change from a local field $L|\mathbb{Q}_p$, then there actually *is* a canonical splitting for every $n \in \mathbb{N}$

$$H^n_{\text{ét}}(X, \mathbb{C}_p) = \bigoplus_{i+j=n} H^i(X, \Omega^j_X(-j)).$$

Tate proves this using Galois cohomology, so this canonical decomposition is an "arithmetic" phenomenon. Here the Tate twist $(-j)$ is really crucial. More generally, such a decomposition is induced by the datum of a lift of X from $\mathbb{C}_p$ to the square-zero extension $\theta : (W(\mathcal{O}_{\mathbb{C}_p^\flat})/(\ker\theta)^2)[\frac{1}{p}] \to \mathbb{C}_p$.

Remark 1.4.34. Note that both sides of (1.6), as well as the terms of (1.7), are algebraic (in the sense that, if X is the analytification of a smooth proper scheme over K, then they are already defined for the scheme), but we think of the isomorphism itself as being analytic or "transcendental" rather than algebraic. In particular, even if both sides are already defined over $\overline{\mathbb{Q}}$, the sequence (1.7) is typically not.

Remark 1.4.35. Comparing Theorem 1.4.30 to the complex decomposition (1.5), you might wonder: What happened to the Kähler condition when we switched from $\mathbb{C}$ to $\mathbb{C}_p$? Answer: It disappeared! Apparently, there is no analogue of the Kähler condition in p-adic geometry, at least not in this context. Maybe one can think of this as a trade-off compared to the complex Hodge decomposition: We win greater generality, but on the other hand we only get a degenerated spectral sequence, not a decomposition (unless e.g. you are in the arithmetic situation of Remark 1.4.33.2).

The final goal of these lectures is to sketch Scholze's proof of Theorem 1.4.30.

[15] In fact, proving finiteness of $H^n_{\text{ét}}(X, \mathbb{C}_p)$ is the first main step in [Sch13a], and it is used in the proof of Theorem 1.4.30, more precisely in the proof of Theorem 1.4.44 below.

1.4.4 The Leray spectral sequence

Building on the work of Faltings, Scholze's idea is to realise the Hodge–Tate spectral sequence in Theorem 1.4.30 as the Leray spectral sequence for the morphism of sites

$$\nu : X_{\text{proét}} \to X_{\text{ét}}$$

and the sheaf $\widehat{O}_X$. This Leray spectral sequence is a priori of the form

$$H^i(X, R^j\nu_*\widehat{O}_X) \Rightarrow H^{i+j}_{\text{proét}}(X, \widehat{O}_X). \tag{1.8}$$

To see that this has the desired form, we need to identify the left hand side with Hodge cohomology and the right hand side with étale cohomology.

The identification with Hodge cohomology works by the following key result:

Theorem 1.4.36. *Let $j \geq 0$. Then there is a canonical and functorial isomorphism*

$$R^j\nu_*\widehat{O}_X = \Omega^j_X(-j).$$

This is quite an amazing result. Note that already the case of $j = 0$ says that $\nu_*\widehat{O} = O$, which is non-trivial, and in fact will give a proof of our earlier Proposition 1.4.15.

Proof. We give a sketch of how Theorem 1.4.36 is proved. There are two main steps:

Step 1. We first show that $R^j\nu_*\widehat{O}$ is finite locally free and that the natural map

$$\textstyle\bigwedge^j R^1\nu_*\widehat{O} \to R^j\nu_*\widehat{O}$$

induced from the cup product in cohomology is an isomorphism. Since $\bigwedge^j \Omega^1_X = \Omega^j_X$ by definition, this will reduce us to identifying the vector bundle $R^1\nu_*\widehat{O}$.

The key idea for the proof of this first step is to use perfectoid covers:

1.4.4.1 Computing pro-étale cohomology, Cartan–Leray style. The statement of Theorem 1.4.36 is local. Since X is smooth, we may therefore without loss of generality assume that $X = \mathrm{Spa}(R, R^+)$ is affinoid and admits an étale morphism to a torus $X \to \mathbb{T}^d$ which is a composition of a rational open immersion with a finite étale map. Recall from Example 1.3.31 and Exercise 1.4.23 that there is a perfectoid pro-étale Galois cover $\mathbb{T}^d_\infty \to \mathbb{T}^d$ with group $\Delta := \mathbb{Z}_p(1)^d$. By pullback, we get a diagram

$$\begin{array}{ccc} X_\infty & \longrightarrow & \mathbb{T}^d_\infty \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathbb{T}^d. \end{array}$$

The following is slightly stronger than what we used in Proposition 1.4.19:

Exercise 1.4.37. Show that X_∞ is an affinoid perfectoid object of $X_{\text{proét}}$, and $X_\infty \to X$ is a pro-finite-étale Galois cover with group $\Delta := \mathbb{Z}_p(1)^d$.

Write $X_\infty = \operatorname{Spa}(R_\infty, R_\infty^+)$. The key computation is now:

Proposition 1.4.38. *There is a natural isomorphism of R-modules*

$$H^n_{\text{proét}}(X, \widehat{\mathcal{O}}_X) \xrightarrow{\sim} H^n_{\text{cts}}(\Delta, R) = \wedge^n R(-1)^d.$$

Proof. This is essentially an application of the "Cartan–Leray spectral sequence":

Lemma 1.4.39 (Cartan–Leray). *Let $X_\infty \to X$ be an affinoid perfectoid object in $X_{\text{proét}}$ that is a Galois cover with pro-finite Galois group G. Then*

$$H^n_{\text{proét}}(X, \widehat{\mathcal{O}}) = H^n_{\text{cts}}(G, \widehat{\mathcal{O}}(X_\infty))$$

Proof. That $X_\infty \to X$ is a Galois cover means by Definition 1.4.22 that

$$X_\infty \times_X X_\infty = \underline{G} \times_{\operatorname{Spa}(K)} X_\infty$$

where $\underline{G}$ is the pro-finite-étale object of $X_{\text{proét}}$ defined in Example 1.4.8. Using Proposition 1.3.39, we deduce from that this that $X_\infty \times_X X_\infty$ is again affinoid perfectoid. Similarly for the higher fibre products of X_∞ over X. We can therefore invoke the pro-étale version of Almost Acyclicity, Theorem 1.4.26, to see that $\widehat{\mathcal{O}}$ is acyclic on the Čech nerve of $X_\infty \to X$. By the Čech-to-sheaf spectral sequence in $X_{\text{proét}}$, we therefore have

$$H^n_{\text{proét}}(X, \widehat{\mathcal{O}}) = \check{H}^n(\{X_\infty \to X\}, \widehat{\mathcal{O}}).$$

Second, we have the following:

Exercise 1.4.40. Let $R_\infty := \widehat{\mathcal{O}}(X_\infty)$. Use the explicit formula for the fibre product in the proof of Proposition 1.3.39 to show that there is a canonical identification

$$\widehat{\mathcal{O}}(X_\infty \times_X X_\infty) = \widehat{\mathcal{O}}(\underline{G} \times X_\infty) = \operatorname{Map}_{\text{cts}}(G, \widehat{\mathcal{O}}(X_\infty)).$$

The same still works for the higher fibre products. It follows that the Čech nerve of $\widehat{\mathcal{O}}$ on $X_\infty \to X$ can be canonically identified with the continuous bar complex of G on $\widehat{\mathcal{O}}(X_\infty)$, which computes continuous group cohomology. Consequently, we have

$$\check{H}^n_{\text{ét}}(\{X_\infty \to X\}, \widehat{\mathcal{O}}) = H^n_{\text{cts}}(G, \widehat{\mathcal{O}}(X_\infty)). \qquad \blacksquare$$

Returning to the proof of Proposition 1.4.38, we are left to compute $H^n_{\text{cts}}(G, \widehat{\mathcal{O}}(X_\infty))$. This is now a fairly standard computation (see [Sch13a, Lemma 5.5] for details). It is deduced by base-change arguments from the following computation:

Exercise 1.4.41 (A bit more challenging[16]). Let $X = \mathbb{T}^1$ so that $R_\infty := \mathcal{O}(X_\infty) = \mathbb{C}_p\langle T^{\pm 1/p^\infty}\rangle$ and $\Delta := \mathbb{Z}_p(1) \cong \mathbb{Z}_p$. Show that in this case, the map

$$R \to H^1_{\mathrm{cts}}(\Delta, R_\infty)$$

is an isomorphism. Hint: first consider the map $R^+/p^n \to H^1_{\mathrm{cts}}(\Delta, R^+_\infty/p^n)$ and use that

$$\mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1/p^\infty}\rangle/p^n = \bigoplus_{m\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}}(\mathcal{O}_{\mathbb{C}_p}/p^n)\cdot T^m$$

as an $\mathcal{O}_{\mathbb{C}_p}$-linear Δ-module. Describe the Δ-action on each factor and its Δ-cohomology. Then apply $\varprojlim_n$ and invert p. If you're stuck, consult [Sch13a, Lemma 5.5]. ■

A computation similar to Exercise 1.4.41 shows that for $X = \mathbb{T}^d$, the map

$$\wedge^n_R R^d = H^n_{\mathrm{cts}}(\Delta, R) \to H^n_{\mathrm{cts}}(\Delta, R_\infty)$$

is an isomorphism, as we wanted to see. In particular, $R^1\nu_*\widehat{\mathcal{O}}$ is a vector bundle.

The identification $R^1\nu_*\widehat{\mathcal{O}} = \Omega^1_X(-1)$ requires further technical input which is outside the scope of these lectures. We give a brief sketch: Following [CBC+19], there is an elegant way to do this via the theory of the derived p-completed cotangent complex $\widehat{L}$. The basic idea is to work on $X_{\text{proét}}$ and consider for any affinoid perfectoid $U \in X_{\text{proét}}$ the transitivity triangle of the composition

$$\mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_p} \to \widehat{\mathcal{O}}^+_X(U).$$

Using that $\widehat{\mathcal{O}}^+_X(U)^a$ is a perfectoid $\mathcal{O}^a_{\mathbb{C}_p}$-algebra, one shows that $\widehat{L}_{\widehat{\mathcal{O}}^+_X(U)|\mathcal{O}_{\mathbb{C}_p}} \overset{a}{=} 0$, and it follows that

$$\widehat{L}_{\widehat{\mathcal{O}}^+_X(U)|\mathbb{Z}_p} \overset{a}{=} \widehat{L}_{\mathcal{O}_{\mathbb{C}_p}|\mathbb{Z}_p} \otimes \widehat{\mathcal{O}}^+_X(U).$$

From results of Tate and Fontaine, it is known that $L_{\mathcal{O}_{\mathbb{C}_p}|\mathbb{Z}_p}[\frac{1}{p}] = \mathbb{C}_p(1)[1]$. On the other hand, there is a natural identification $\Omega^1_X = \widehat{L}_{\mathcal{O}_X|\mathcal{O}_{\mathbb{C}_p}}[\frac{1}{p}]$ on $X_{\text{ét}}$. Putting everything together, we obtain the desired map on $X_{\text{proét}}$

$$\nu^*\Omega^1_X \to \nu^*\widehat{L}_{\mathcal{O}_X|\mathcal{O}_{\mathbb{C}_p}}[\tfrac{1}{p}] \to \widehat{L}_{\widehat{\mathcal{O}}_X|\mathcal{O}_{\mathbb{C}_p}}[\tfrac{1}{p}] \to \widehat{\mathcal{O}}_X(1)[1]$$

Forming the pushforward along $\nu : X_{\text{proét}} \to X_{\text{ét}}$, we get a natural morphism

$$\Omega^1_X \to R^1\nu_*\widehat{\mathcal{O}}_X(1)$$

which gives the desired morphism by twisting with $\mathbb{Z}_p(-1)$. ■

[16] But the kind of computation that you may find enlightening when you work it out yourself.

This finishes the proof of Theorem 1.4.36. For readers who would like to get some practice with the ideas introduced in the proof, we offer the following exercise:

Exercise 1.4.42 (Even more challenging). Show that $H^2_{\text{ét}}(\mathbb{T}^1, \mathcal{O}^+)$ has unbounded p-torsion, by following the sketch below:

(1) Use a long exact sequence to reduce to computing $H^1_{\text{ét}}(\mathbb{T}^1, \mathcal{O}/\mathcal{O}^+)$. For this you may use that $H^i_{\text{ét}}(\mathbb{T}^1, \mathcal{O}) = 0$ for $i \geq 1$ (see [FvdP04, Proposition 8.2.3]).

(2) Convince yourself that Lemma 1.4.39 also holds with $\widehat{\mathcal{O}}$ replaced by $\mathcal{O}/\mathcal{O}^+$.

(3) Use Lemma 1.4.17 and the pro-étale cover $\mathbb{T}^1_\infty \to \mathbb{T}^1$ to see that $H^1_{\text{ét}}(\mathbb{T}^1, \mathcal{O}/\mathcal{O}^+) = H^1_{\text{proét}}(\mathbb{T}^1, \mathcal{O}/\mathcal{O}^+)$. (This holds more generally, see [Sch13a, Corollary 3.17.(i)].)

(4) Use an integral version of Exercise 1.4.41 to show that there is a natural map $\mathcal{O}^+(R^+) \to H^1_{\text{proét}}(\mathbb{T}^1, \mathcal{O}^+)$ whose cokernel is killed by p.

(5) Deduce: $H^1_{\text{proét}}(\mathbb{T}^1, \mathcal{O}/\mathcal{O}^+) = H^1_{\text{proét}}(\mathbb{T}^1, \mathcal{O}^+) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ has unbounded p-torsion.

Using additionally [Sch13a, Lemma 4.5], one can show in the same way that also $H^2_{\text{ét}}(\mathbb{B}^1, \mathcal{O}^+)$ has unbounded p-torsion. This was mentioned in Remark 1.3.48.

1.4.4.2 Aside: Relation to Galois cohomology. Computations resembling the one in this subsection have played a role in p-adic Hodge theory long before perfectoid spaces where introduced. But even for these older computations, the pro-étale site often gives a nice new geometric perspective that helps understand conceptually what is going on. For example, in Tate's article [Tat67] on the Hodge–Tate comparison for abelian varieties, there is a computation of the Galois cohomology of $H^1_{\text{cts}}(G_{\mathbb{Q}_p}, \mathbb{C}_p)$ which in this newer language can be interpreted as follows:

Let $L \subseteq \mathbb{C}_p$ be any finite extension of $\mathbb{Q}_p$. Then $X = \mathrm{Spa}(L, \mathcal{O}_L)$ is a rigid space.

Exercise 1.4.43. Show that we can regard $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ as a perfectoid object in $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})_{\text{proét}}$, namely as the limit $\widehat{U}$ of the cofiltered inverse system U of the finite subextensions $L \subseteq L' \subseteq \mathbb{C}_p$.

By a common abuse of notation, let us ignore the difference between $\widehat{U}$ and the inverse system U, then we can regard $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to \mathrm{Spa}(L, \mathcal{O}_L)$ as a Galois cover with group $G_L := \mathrm{Gal}(\overline{L}|L)$ in the sense of Definition 1.4.22. We can now again use the Cartan–Leray spectral sequence, Lemma 1.4.39, and compute that

$$H^1_{\text{proét}}(\mathrm{Spa}(L), \widehat{\mathcal{O}}) = H^1_{\text{cts}}(G_L, \mathbb{C}_p).$$

On the other hand, we know from lecture 1 that there are much smaller perfectoid Galois covers of L: Namely, assume for simplicity that $\zeta_p \in L$ and let $L^{\text{cyc}}|L$ be the compositum of $\mathbb{Q}_p^{\text{cyc}}|\mathbb{Q}_p$ and $L|\mathbb{Q}_p$, then $\mathrm{Spa}(L^{\text{cyc}}) \to \mathrm{Spa}(L)$, is an affinoid perfectoid Galois cover with group $\mathbb{Z}_p$. Again by Lemma 1.4.39, we thus have

$$H^1_{\text{proét}}(\mathrm{Spa}(L), \widehat{\mathcal{O}}) = H^1_{\text{cts}}(\mathbb{Z}_p, L^{\text{cyc}}).$$

By a direct calculation, Tate shows that the right hand side is $= L$. All in all, this shows that

$$H^1_{\mathrm{cts}}(G_L, \mathbb{C}_p) = L.$$

Note that this also shows that $H^1_{\text{proét}}(\mathrm{Spa}(L), \widehat{\mathcal{O}}) = L$, which underlines how special it is that $\widehat{\mathcal{O}}$ is acyclic on perfectoid spaces: Indeed, more generally speaking, for any non-archimedean field K over $\mathbb{Q}_p$, sheaf cohomology of $\mathrm{Spa}(K)_{\text{proét}}$ is essentially the same as Galois cohomology of K, and the translation is given by considering the pro-étale Galois cover $\mathrm{Spa}(C) \to \mathrm{Spa}(K)$ with group G_K, where C is the completion of an algebraic closure of K. Both perspectives have advantages: For example, sheaf cohomology on $\mathrm{Spa}(K)_{\text{proét}}$ is defined by an actual derived functor, while Galois cohomology is not, being defined rather explicitly via the bar complex. On the other hand, in practice one can often use the concrete definition of continuous group cohomology in order to actually compute sheaf cohomology “by hand”.

1.4.5 Step 2: The Primitive Comparison Theorem

Given Theorem 1.4.36, the Leray spectral sequence (1.8) for $\nu : X_{\text{proét}} \to X_{\text{ét}}$ now has the form

$$H^i_{\text{ét}}(X, \Omega^j_X(-j)) \Rightarrow H^{i+j}_{\text{proét}}(X, \widehat{\mathcal{O}}).$$

In order to deduce Theorem 1.4.30, we are left to identify $H^{i+j}_{\text{proét}}(X, \widehat{\mathcal{O}})$ with étale cohomology. Scholze’s second step is therefore to prove the following comparison result, which he calls the “Primitive Comparison Theorem”:

Theorem 1.4.44 ([Sch13a, Theorem 5.1]). *Let X be any smooth proper rigid space over $\mathbb{C}_p$. Then the following natural map is an isomorphism:*

$$H^n_{\text{ét}}(X, \mathbb{C}_p) \xrightarrow{\sim} H^n_{\text{proét}}(X, \widehat{\mathcal{O}})$$

A related result had previously been proved by Faltings [Fal02, §3, Theorem 8]. The proof is quite difficult, and ultimately relies on a tilting argument which reduces to a statement in characteristic p about the Artin–Schreier sequence (believe it or not).

1.4.5.1 The case of abelian varieties. The proof of Theorem 1.4.44 is well beyond the scope of these lectures. Apart from the above extremely short summary, we sketch a proof of it in the special case when X is an abelian variety of good reduction. This is accessible with the methods we have already developed. Once again, this follows an idea due to Bhargav Bhatt from his Arizona Winter School lectures in [CBC+19].

Remark 1.4.45. The case of abelian varieties of Theorem 1.4.30 has itself a long and influential history: As we already mentioned, the first instance was proved by Tate [Tat67, Remark on p180] for abelian varieties of good reduction over a discrete valuation ring and it let him to ask if (1.7) holds more generally. The good reduction assumption was subsequently removed by Raynaud based on the theory of semistable reduction. Fontaine gave a different proof in [Fon82, Théorème 2] via a "p-adic integration map", a perspective that was further developed by Coleman [Col84].

Let $X = A$ be an abelian variety over $\mathbb{C}_p$ of good reduction, considered as an adic space. Recall from Example 1.4.24 the **p-adic universal cover**

$$A_\infty = \varprojlim_{[p]} A \to A$$

in $A_{\text{proét}}$. The name originates from the theory of p-divisible groups [SW13], but in fact A_∞ does behave quite a bit like a topological universal cover! Namely, we have the following weaker analogue of the topological universal cover being simply connected:

Proposition 1.4.46 ([CBC+19, Bhatt, Proposition 2.1.1]).

$$(1)\quad H^n_{\text{proét}}(A_\infty, \widehat{\mathcal{O}}) \stackrel{a}{=} \begin{cases} \mathbb{C}_p & n = 0, \\ 0 & n > 0, \end{cases} \qquad (2)\quad H^n_{\text{proét}}(A_\infty, \underline{\mathbb{C}}_p) \stackrel{a}{=} \begin{cases} \mathbb{C}_p & n = 0, \\ 0 & n > 0. \end{cases}$$

Remark 1.4.47. This is still true without the good reduction hypothesis on A, see [Heu21, Proposition 4.2]. However, the proof in the given reference uses the Primitive Comparison Theorem, so it would be cyclical to use this to prove Theorem 1.4.44. That being said, there is an alternative way to prove Proposition 1.4.46 for general A without good reduction assumption, so with a little bit more work the good reduction assumption can be removed while still following the same line of argument.

By a similar argument as in Lemma 1.4.39, it follows that $H^n_{\text{proét}}(A, \widehat{\mathcal{O}})$ can be computed via the Cartan–Leray spectral sequence. Applying this to $\widehat{\mathcal{O}}$ yields

$$H^n_{\text{proét}}(A, \widehat{\mathcal{O}}) = H^n_{\text{cts}}(T_pA, \mathbb{C}_p),$$

By Proposition 1.4.46.(2), there is an analogous Cartan–Leray sequence for $\underline{\mathbb{C}}_p$:

$$H^n_{\text{proét}}(A, \underline{\mathbb{C}}_p) = H^n_{\text{cts}}(T_pA, \mathbb{C}_p).$$

Comparing the two gives the desired isomorphism.

This finishes the proof of Theorem 1.4.30 for abelian varieties of good reduction.

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