

DIAMANTINE PICARD FUNCTORS OF RIGID SPACES

ABSTRACT. For a connected smooth proper rigid space X over a perfectoid field extension of \mathbb{Q}_p , we show that the étale Picard functor of X^\diamond defined on perfectoid test objects is the diamondification of the rigid analytic Picard functor. In particular, it is represented by a rigid analytic group variety if and only if the rigid analytic Picard functor is.

Second, we study the v -Picard functor that parametrises line bundles in the finer v -topology on the diamond associated to X and relate this to the rigid analytic Picard functor by a geometrisation of the multiplicative Hodge–Tate sequence.

The motivation is an application to the p -adic Simpson correspondence, namely our results pave the way towards the first instance of a new moduli theoretic perspective.

1. INTRODUCTION

Line bundles are ubiquitous in rigid analytic geometry, and Picard groups of rigid spaces have therefore been the subject of extensive studies, for example in [9, 2, 10, 4, 26, 22, 24]. It is natural to ask how much of this theory carries over to Scholze’s larger categories like the pro-étale site [30] or the category of diamonds [32]. One particular question with a long history in rigid geometry is about the existence of rigid analytic moduli spaces of line bundles on proper rigid spaces, namely about representability of the rigid Picard functor which parametrises isomorphism classes of line bundles:

Let p be a prime, let K be a perfectoid extension of \mathbb{Q}_p and let $\pi : X \rightarrow \mathrm{Spa}(K)$ be a smooth proper rigid space considered as an adic space. The rigid Picard functor is the sheaf

$$\mathbf{Pic}_{X,\acute{e}t} := R^1 \pi_{\acute{e}t*} \mathbb{G}_m : \mathrm{SmRig}_{K,\acute{e}t} \rightarrow \mathrm{Ab}$$

where $\mathrm{SmRig}_{K,\acute{e}t}$ is the big étale site of smooth rigid spaces over K and $\pi_{\acute{e}t*}$ is the pushforward of big étale sites along π . Explicitly, $\mathbf{Pic}_{X,\acute{e}t}$ is the sheafification of the functor sending a smooth rigid space Y to the group of isomorphism classes of line bundles on $X \times Y$. It is expected that $\mathbf{Pic}_{X,\acute{e}t}$ is always representable, and this is known in many cases of interest:

1. If X is the analytification of a smooth proper algebraic variety X_0 , then it follows from Köpf’s relative rigid GAGA-Theorem [23][25, Theorem 2.8][4, Example 3.2.6] that $\mathbf{Pic}_{X,\acute{e}t}$ is the analytification of the algebraic Picard variety of X_0 [2, §1]. In particular, the identity component $\mathbf{Pic}_{X,\acute{e}t}^0$ is then an abelian variety.
2. In [3, §6], Bosch–Lütkebohmert treated the case that X is an abeloid variety, i.e. a connected smooth proper rigid group variety: $\mathbf{Pic}_{X,\acute{e}t}^0$ is then the dual abeloid variety.
3. Hartl–Lütkebohmert [10] proved that if X has a strict semi-stable formal model over a discrete valuation ring, then $\mathbf{Pic}_{X,\acute{e}t}$ is represented by a rigid group such that $\mathbf{Pic}_{X,\acute{e}t}^0$ is semi-abeloid, i.e. an extension of an abeloid variety by a torus.
4. Warner has announced in his thesis [36] a proof that $\mathbf{Pic}_{X,\acute{e}t}$ is always representable, but he does not describe what $\mathbf{Pic}_{X,\acute{e}t}^0$ looks like in general.

1.1. The étale diamantine Picard functor. The goal of this article is to study Picard functors that are instead defined on perfectoid spaces as defined by Scholze [29]. Viewing

rigid spaces through perfectoid spaces naturally leads us into the setting of diamonds introduced in [32]: Let $\mathrm{Perf}_{K,\acute{\mathrm{e}}\mathrm{t}}$ be the site of perfectoid spaces over X equipped with the étale topology, and $\mathrm{Perf}_{K,v}$ the same category with the much finer v -topology from [32, §8]. Associated to X we have the diamondification

$$\pi^\diamond : X^\diamond \rightarrow \mathrm{Spd}(K)$$

defined in [32, §15]. This is a morphism of diamonds, and thus of sheaves on $\mathrm{Perf}_{K,v}$. Analogously to the rigid case, we define the étale diamantine Picard functor to be the sheaf

$$\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}^\diamond := R^1\pi_{\acute{\mathrm{e}}\mathrm{t}*}^\diamond \mathbb{G}_m : \mathrm{Perf}_{K,\acute{\mathrm{e}}\mathrm{t}} \rightarrow \mathrm{Ab}.$$

Explicitly, this is the étale sheafification of the functor that sends a perfectoid space Y over K to the set of isomorphism classes of line bundles on the analytic adic space $X \times Y$.

The first main result of this article is that this new diamantine Picard functor $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}^\diamond$ can be described in terms of the rigid analytic Picard functor $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}$. For this introduction, let us for simplicity assume that $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}$ is represented by a rigid space. Then we show:

Theorem 1.1. *There is a natural isomorphism of sheaves on $\mathrm{Perf}_{K,\acute{\mathrm{e}}\mathrm{t}}$*

$$(\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}})^\diamond \xrightarrow{\sim} \mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}^\diamond,$$

that is, $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}^\diamond$ is represented by the diamondification of the rigid analytic Picard functor.

Corollary 1.2. *If X is connected and $x \in X(K)$ is a point, then for any perfectoid space T over K , the isomorphism classes of line bundles on $X \times T$ that are trivial over $x \times T$ are in natural one-to-one correspondence with the morphisms of adic spaces $T \rightarrow \mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}$ over K .*

In fact, we define more generally a natural “diamondification of functors on $\mathrm{SmRig}_{K,\acute{\mathrm{e}}\mathrm{t}}$ ” such that the statement of Theorem 1.1 holds without requiring $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}$ to be representable: Explicitly, the result then says that for every perfectoid space T and any line bundle L on $X \times T$, one can étale-locally on T find a rigid space T_0 and a morphism $T \rightarrow T_0$ such that L descends to $X \times T_0$. This statement is not at all obvious, and it is false in general if we remove the assumption that X is proper.

1.2. The v -Picard functor. In the diamantine setting, there is now also a second natural Picard functor

$$\mathbf{Pic}_{X,v}^\diamond := R^1\pi_{v*}^\diamond \mathbb{G}_m : \mathrm{Perf}_{K,v} \rightarrow \mathrm{Ab},$$

given by the v -sheafification of the functor that sends Y to the sheaf of isomorphism classes of v -line bundles on $X \times Y$. This difference in topology is more than just a technicality: We showed in [14, Theorem 1.3.2a] that there are in general many more v -line bundles on X than étale line bundles. In fact, we showed that if K is algebraically closed, the respective Picard groups of isomorphism classes fit into a “multiplicative Hodge–Tate sequence”:

$$0 \rightarrow \mathrm{Pic}_{\acute{\mathrm{e}}\mathrm{t}}(X) \rightarrow \mathrm{Pic}_v(X) \rightarrow H^0(X, \tilde{\Omega}^1) \rightarrow 0$$

where $\tilde{\Omega}^1 := \Omega^1\{-1\}$ is a Tate twist. Our second main result is that this short exact sequence can be geometrised to give a comparison between the étale and v -topological Picard functors:

Theorem 1.3. *Let X be a smooth proper rigid space over a perfectoid extension K of \mathbb{Q}_p . Then the v -Picard functor fits into a natural exact sequence of abelian sheaves on $\mathrm{Perf}_{K,\acute{\mathrm{e}}\mathrm{t}}$*

$$0 \rightarrow \mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}^\diamond \rightarrow \mathbf{Pic}_{X,v}^\diamond \rightarrow H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a \rightarrow 0.$$

In particular, $\mathbf{Pic}_{X,v}^\diamond$ is represented by a rigid group variety if and only if $\mathbf{Pic}_{X,\acute{\mathrm{e}}\mathrm{t}}$ is.

One consequence is that $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ is in fact a v -sheaf, so we can regard $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ as an extension of $\mathbf{Pic}_{X,\acute{e}t}$ to a much larger class of test objects. Conversely, we show that if $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ is represented by a rigid group, then this also represents $\mathbf{Pic}_{X,\acute{e}t}$. As we explain in more detail below, this might open up new methods to study $\mathbf{Pic}_{X,\acute{e}t}$ and its representability.

1.3. Applications to non-abelian p -adic Hodge theory. Theorem 1.1 and Theorem 1.3 are of interest in their own right, but our main motivation for studying diamantine Picard varieties is an application to the p -adic Simpson correspondence:

The reason why perfectoid test objects appear naturally in this context is that in [13] we describe a class of “topological torsion” line bundles L on X characterised by the property that L extends to a line bundle on the adic space $X \times \widehat{\mathbb{Z}}$ such that the specialisation of L at $n \in \mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ is isomorphic to L^n . We would like these to be precisely those line bundles which induce a homomorphism of adic groups $\widehat{\mathbb{Z}} \rightarrow \mathbf{Pic}_X$. This is guaranteed by Theorem 1.1.

While this was our original motivation to study diamantine Picard functors, Theorem 1.3 exhibits a much deeper connection, namely it paves the way for a new understanding of the p -adic Simpson correspondence as a geometric comparison of moduli spaces: The short exact sequence in Theorem 1.3 gives a geometrisation of the equivalence between v -line bundles and Higgs line bundles of [14, §5]. Geometrically speaking, it says that there is a rigid analytic moduli space of v -line bundles $\mathbf{Pic}_{X,v}^\diamond$ which is a $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ -torsor over $\mathcal{A} := H^0(X, \widetilde{\Omega}_X^1) \otimes_K \mathbb{G}_a$ in a natural way. As the same is true for the moduli space $\mathbf{Higgs}_1 := \mathbf{Pic}_{X,\acute{e}t}^\diamond \times \mathcal{A}$ of Higgs line bundles, we find that $\mathbf{Pic}_{X,v}^\diamond$ is an étale twist of \mathbf{Higgs}_1 over \mathcal{A} . As we explain in detail in the sequel [13], this leads to a geometric non-abelian Hodge correspondence in rank one.

The present article is therefore the first in a series on the geometrisation of the p -adic Simpson correspondence. Indeed, in [18], we will show that when X is a curve, the perspective that the moduli space of v -vector bundles is a twist of the moduli space of Higgs bundles generalises to objects of higher rank.

1.4. The topological torsion Picard functor is representable. Given the known results about representability of the rigid Picard functor which we summarised above, it seems plausible that $\mathbf{Pic}_{X,\acute{e}t}$ is always representable by a rigid group variety whose identity component is a semi-abeloid variety.

In order to illustrate how the perfectoid perspective can help understand the structure of Picard functors, let us already mention the following result from the sequel article [13], saying that a topological torsion version of the rigid Picard functor is always representable: Let \mathbb{G}_m be the subgroup of topologically p -torsion units, given by the open disc at 1 of radius 1. We then define the topologically p -torsion Picard functor of X to be

$$\widehat{\mathbf{Pic}}_{X,\acute{e}t} := R^1 \pi_{\acute{e}t*} \widehat{\mathbb{G}}_m : \mathrm{SmRig}_{K,\acute{e}t} \rightarrow \mathrm{Ab}$$

Using a geometric p -adic Simpson correspondence in terms of $\mathbf{Pic}_{X,\acute{e}t}^\diamond$, we prove in [13] that $\widehat{\mathbf{Pic}}_{X,\acute{e}t}$ is always representable by a finite disjoint union of analytic p -divisible groups in the sense of Fargues [6, §1.6]. As such, it is a subgroup of $\underline{\mathrm{Hom}}(\pi_1(X), \widehat{\mathbb{G}}_m)$ where $\pi_1(X)$ is the étale fundamental group of X . If $\mathbf{Pic}_{X,\acute{e}t}$ is represented by a rigid group G , then this is the topological p -torsion subgroup of G as defined by Fargues [6, §1.6].

This produces some evidence that $\mathbf{Pic}_{X,\acute{e}t}$ is always representable by a rigid group whose connected component is a semi-abeloid variety: Namely, it imposes restrictions on what kind of rigid groups can appear as Picard varieties, which are consistent with this prediction.

Acknowledgements. We thank Johannes Anschütz, Urs Hartl, Lucas Mann, Alexander Petrov, Peter Scholze, Annette Werner, Daxin Xu and Bogdan Zavyalov for very useful conversations.

We thank Lucas Gerth for pointing out several mistakes in an earlier version. Special thanks to the referee for many helpful comments which have greatly improved the article.

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy– EXC-2047/1 – 390685813. The author was supported by the DFG via the Leibniz-Preis of Peter Scholze.

Notation. Let K a complete non-archimedean field of residue characteristic p . Let \mathcal{O}_K be the ring of integers of K and let $K^+ \subseteq \mathcal{O}_K$ be any ring of integral elements. Let $\mathfrak{m} \subseteq K^+$ be the ideal of topologically nilpotent elements and fix a pseudo-uniformiser $\varpi \in \mathfrak{m}$. We use adic spaces in the sense of Huber [19]. We abbreviate $\mathrm{Spa}(K, K^+)$ by $\mathrm{Spa}(K)$ when K^+ is clear from the context.

In this article, a rigid space over K is by definition an adic space locally of topologically finite type over $\mathrm{Spa}(K, K^+)$. In particular, we then denote by SmRig_K the category of smooth rigid spaces over $\mathrm{Spa}(K, K^+)$. For any rigid space X we denote by SmRig_X the slice category of rigid spaces over X . For rigid spaces X, Y over K , we also write $Y_X := Y \times_K X$. We denote by \mathbb{B}^d the unit disc of dimension d over K , considered as an affinoid rigid space.

We are most interested in the case of $K^+ = \mathcal{O}_K$, in which case the notion of rigid spaces is equivalent to the classical one under mild technical assumptions, but like in [30] it is useful to consider the more general case since this makes it very easy to later pass to the relative situation of morphisms of rigid spaces $\pi : X \rightarrow Y$, which is useful for applications.

We use perfectoid spaces in the sense of [29] and denote by Perf_K the category of perfectoid spaces over (K, K^+) . We use diamonds in the sense of [32] and denote by LSD_K the category of locally spatial diamonds over $\mathrm{Spd}(K, K^+)$. For any $X \in \mathrm{LSD}_K$, we denote by LSD_X the slice category.

By a rigid group over K we mean a group object in the category of rigid spaces over K , always assumed to be commutative. We denote by \mathbb{G}_m the rigid multiplicative group, by \mathbb{G}_a the rigid additive group, and by $\mathbb{G}_a^+ \subseteq \mathbb{G}_a$ the subgroup given by the closed unit ball \mathbb{B}^1 .

2. THE DIAMANTINE PICARD FUNCTORS

Let K be a perfectoid field over \mathbb{Q}_p and let $\pi : X \rightarrow \mathrm{Spa}(K, K^+)$ be a smooth rigid space. The aim of this section is to introduce the diamantine Picard functors of X , and to state a more precise and more general version of the main results, as well as some corollaries. We do not give any proofs yet, but we end the section with an overview of the strategy of proof.

2.1. Definition of diamantine and rigid Picard functors. Recall that the rigid analytic Picard functor as considered in [10] can be defined as the abelian sheaf

$$\mathbf{Pic}_{X, \acute{e}t} := R^1 \pi_{\acute{e}t*} \mathbb{G}_m : \mathrm{SmRig}_{K, \acute{e}t} \rightarrow \mathrm{Ab}$$

where $\mathrm{SmRig}_{K, \acute{e}t}$ is the site of smooth rigid spaces over $\mathrm{Spa}(K, K^+)$ with the étale topology and where $\pi_{\acute{e}t} : \mathrm{SmRig}_{X, \acute{e}t} \rightarrow \mathrm{SmRig}_{K, \acute{e}t}$ is the natural morphism of big étale sites. Conjecturally, if X is proper, then $\mathbf{Pic}_{X, \acute{e}t}$ is represented by a smooth rigid group variety. As summarised in the introduction, this is known in many cases, but not yet in full generality.

Our goal in this subsection is to introduce a “diamantine” variant of the Picard functor defined on perfectoid test objects, and to explain how this can be compared to $\mathbf{Pic}_{X, \acute{e}t}$.

Recall from [32, §15] that there is a fully faithful diamondification functor

$$\mathrm{SmRig}_K \rightarrow \mathrm{LSD}_K, \quad X \mapsto X^\diamond$$

sending a smooth rigid space X to its associated locally spatial diamond over $\mathrm{Spd}(K, K^+)$. We sometimes drop $-\diamond$ from notation when this is clear from the context, for example we

simply write \mathbb{G}_m for the diamond that sends a perfectoid space $\mathrm{Spa}(R, R^+)$ to R^\times . We write δ for the morphism of étale sites associated to the above functor (cf [32, Lemma 15.6])

$$\delta : \mathrm{LSD}_{K, \acute{e}t} \rightarrow \mathrm{SmRig}_{K, \acute{e}t}.$$

We also need the analogous functor on the site of perfectoid spaces with the étale topology:

$$\iota : \mathrm{LSD}_{K, \acute{e}t} \rightarrow \mathrm{Perf}_{K, \acute{e}t}.$$

Since for any adic space Y over K we have $Y_{\acute{e}t} \cong Y_{\acute{e}t}^\diamond$, both δ_* and ι_* are exact functors.

For the definition of the diamantine Picard functor, we now consider the diamondification $\pi^\diamond : X^\diamond \rightarrow \mathrm{Spd}(K, K^+)$. Pullback along this map induces a natural morphism of sites

$$\pi_{\acute{e}t}^\diamond : \mathrm{LSD}_{X, \acute{e}t} \rightarrow \mathrm{Perf}_{K, \acute{e}t}.$$

There is a second, much finer topology on perfectoid spaces and diamonds, namely the v -topology introduced by Scholze in [32, §8, §14]. For this we get a morphism of sites

$$\pi_v^\diamond : \mathrm{LSD}_{X, v} \rightarrow \mathrm{Perf}_{K, v}$$

with the same underlying functor as $\pi_{\acute{e}t}^\diamond$, but finer topologies on either side.

Definition 2.1. The étale diamantine Picard functor of X is defined to be the sheaf

$$\mathbf{Pic}_{X, \acute{e}t}^\diamond := R^1 \pi_{\acute{e}t*}^\diamond \mathbb{G}_m : \mathrm{Perf}_{K, \acute{e}t} \rightarrow \mathrm{Ab}.$$

In the diamantine setting, there is a second Picard functor defined using the finer v -topology:

$$\mathbf{Pic}_{X, v}^\diamond := R^1 \pi_{v*}^\diamond \mathbb{G}_m : \mathrm{Perf}_{K, v} \rightarrow \mathrm{Ab}.$$

Remark 2.2. The functor $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ naturally extends to $\mathrm{LSD}_{X, \acute{e}t}$ if we instead define $\pi_{\acute{e}t}^\diamond$ to be the morphism $\mathrm{LSD}_{X, \acute{e}t} \rightarrow \mathrm{LSD}_{K, \acute{e}t}$. We can then equivalently define $\mathbf{Pic}_{X, \acute{e}t}^\diamond := \iota_* R^1 \pi_{\acute{e}t*}^\diamond \mathbb{G}_m$. The same works for $\mathbf{Pic}_{X, v}^\diamond$, here the difference is less relevant as $\mathrm{Perf}_{K, v}$ is a basis of $\mathrm{LSD}_{K, v}$.

That said, for our purposes it is important to restrict to perfectoid test objects: One reason is that \mathbb{G}_m is easier to describe on adic spaces than on diamonds, another that relative p -adic Hodge theory is much simpler for morphisms $X \times Y \rightarrow Y$ over a perfectoid base Y .

The two functors $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ and $\mathbf{Pic}_{X, v}^\diamond$ are related via a natural morphism: One way to construct this is via the Leray sequences for the compositions in the commutative diagram

$$\begin{array}{ccc} \mathrm{LSD}_{X^\diamond, v} & \xrightarrow{\pi_v^\diamond} & \mathrm{Perf}_{K, v} \\ \downarrow & \searrow \text{dotted} & \downarrow \nu \\ \mathrm{LSD}_{X^\diamond, \acute{e}t} & \xrightarrow{\pi_{\acute{e}t}^\diamond} & \mathrm{Perf}_{K, \acute{e}t}. \end{array}$$

These induce a natural map

$$\mathbf{Pic}_{X, \acute{e}t}^\diamond \rightarrow \nu_* \mathbf{Pic}_{X, v}^\diamond.$$

Since both sides have the same underlying category Perf_K , we shall in the following drop ν_* from notation, which amounts to forgetting that $\mathbf{Pic}_{X, v}^\diamond$ is already a sheaf for the v -topology.

Second, the two functors $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ and $\mathbf{Pic}_{X, v}^\diamond$ are related to the rigid analytic Picard functor via a natural base-change map: Consider the commutative diagram of big étale sites

$$\begin{array}{ccc} \mathrm{LSD}_{X^\diamond, \acute{e}t} & \xrightarrow{\delta} & \mathrm{SmRig}_{X, \acute{e}t} \\ \downarrow & & \downarrow \pi_{\acute{e}t} \\ \mathrm{LSD}_{K, \acute{e}t} & \xrightarrow{\delta} & \mathrm{SmRig}_{K, \acute{e}t}. \end{array}$$

For the comparison, we wish to extend the diamondification functor from smooth rigid spaces to sheaves on $\mathrm{SmRig}_{K,\acute{e}t}$. For any sheaf \mathcal{F} on $\mathrm{SmRig}_{K,\acute{e}t}$, we therefore define

$$\mathcal{F}^\diamond := \iota_* \delta^{-1} \mathcal{F}.$$

This notation makes sense since \mathcal{F}^\diamond agrees with X^\diamond if \mathcal{F} is represented by a smooth rigid space X . We note that $-^\diamond$ is exact because ι_* is. Consequently, the base change map for the above diagram induces for any $n \geq 0$ a natural morphism of sheaves on $\mathrm{Perf}_{K,\acute{e}t}$

$$(1) \quad (R^n \pi_{\acute{e}t*} \mathcal{F})^\diamond \rightarrow R^n \pi_{\acute{e}t*}^\diamond \mathcal{F}^\diamond.$$

Applying this to $\mathcal{F} = \mathbb{G}_m$ and $n = 1$, we find that there is a natural morphism

$$(\mathbf{Pic}_{X,\acute{e}t})^\diamond \rightarrow \mathbf{Pic}_{X,\acute{e}t}^\diamond$$

of sheaves on $\mathrm{Perf}_{K,\acute{e}t}$. It is clear from the construction that this is functorial in X .

2.2. The Diamantine Picard Comparison Theorem. With the technical preparations of the last section, we can now formulate the most general version of our main result. As this can be interpreted as being “relative p -adic Hodge theory for \mathbb{G}_m ”, we first explain a related but simpler result, namely the case of \mathbb{G}_a .

To simplify notation in the following without making additional choices, we begin by introducing notation for the usual Tate twist in p -adic Hodge theory:

Definition 2.3. For any smooth rigid space X over K , we denote by $\tilde{\Omega}_X^1 := \Omega_{X|K}^1\{-1\}$ the sheaf on $X_{\acute{e}t}$ given by tensoring over K with the Breuil–Kisin–Fargues twist $K\{-1\}$. If K contains all p -power unit roots, this twist is equivalent to the usual Tate twist $K(-1)$.

Assume from now on that X is proper. As we will discuss, following Scholze’s approach to the Hodge–Tate sequence of p -adic Hodge theory [30][31, §3], one has in general:

Proposition 2.4. *There is a natural short exact sequence*

$$0 \rightarrow H_{\mathrm{an}}^1(X, \mathcal{O}) \rightarrow H_v^1(X, \mathcal{O}) \rightarrow H^0(X, \tilde{\Omega}_X^1) \rightarrow 0.$$

This is canonically isomorphic to the usual Hodge–Tate short exact sequence if K is algebraically closed, via the Primitive Comparison Theorem: $H_v^1(X, \mathcal{O}) = H_{\acute{e}t}^1(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K$.

As we will show, due to various base-change results for coherent cohomology, we have a relative “diamantine” version of this short exact sequence. For the formulation, we need:

Definition 2.5. For any abelian sheaf F on $\mathrm{Perf}_{K,\acute{e}t}$, we define its tangent space as

$$T_0 F := \mathrm{Hom}(\mathbb{G}_a^+, F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where \mathbb{G}_a^+ is the closed unit with its additive structure. Since $\mathrm{Hom}(\mathbb{G}_a^+, \mathbb{G}_a^+) = K^+$, this is always a K -vector space. When F is represented by a rigid group variety G , this recovers the usual notion of tangent spaces by [12, Theorem 3.4].

Proposition 2.6. *Let (K, K^+) be a perfectoid field extension of \mathbb{Q}_p . Let $\pi : X \rightarrow \mathrm{Spa}(K, K^+)$ be any proper smooth rigid space. Then:*

1. *The natural map $(R^1 \pi_{\acute{e}t*} \mathcal{O})^\diamond \rightarrow R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}$ from (1) is an isomorphism.*
2. *There is a short exact sequence of sheaves on $\mathrm{Perf}_{K,\acute{e}t}$, functorial in X ,*

$$(2) \quad 0 \rightarrow R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O} \rightarrow R^1 \pi_{v*}^\diamond \mathcal{O} \xrightarrow{\mathrm{HT}} H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a \rightarrow 0$$

which is canonically isomorphic to $- \otimes_K \mathbb{G}_a$ applied to the sequence in Proposition 2.4. In particular, we recover this sequence by passing to tangent spaces.

We note that if K is algebraically closed, this shows that $R^1 \pi_{v*}^\diamond \mathcal{O} = \mathrm{Hom}(\pi_1(X), K) \otimes \mathbb{G}_a$.

We can now formulate a precise version of the main result of this article, which could be described as a version of Proposition 2.6 for the multiplicative group \mathbb{G}_m .

Theorem 2.7 (Diamantine Picard Comparison Theorem). *Let (K, K^+) be a perfectoid field extension of \mathbb{Q}_p and let $X \rightarrow \mathrm{Spa}(K, K^+)$ be any proper smooth rigid space. Then:*

1. *The natural map $(\mathbf{Pic}_{X, \acute{e}t})^\diamond \rightarrow \mathbf{Pic}_{X, \acute{e}t}^\diamond$ is an isomorphism.*
 2. *There is a short exact sequence of abelian sheaves on $\mathrm{Perf}_{K, \acute{e}t}$, functorial in X ,*
- $$(3) \quad 0 \rightarrow \mathbf{Pic}_{X, \acute{e}t}^\diamond \rightarrow \mathbf{Pic}_{X, v}^\diamond \xrightarrow{\mathrm{HT} \log} H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a \rightarrow 0.$$

On tangent spaces at the identity, this recovers the sequence (2).

3. *The sequence becomes split over the bounded open subgroup of $H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a$ defined as the image of $H_v^1(X, \mathcal{O}^+) \otimes p\mathbb{G}_a^+$ under the Hodge–Tate map HT from (2).*

Remark 2.8. We will see in [13] that the sequence (3) is never split globally over all of $H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a$ except in the trivial case of $H^0(X, \Omega_X^1) = 0$. In fact, it is better to think about the morphism HT log as a non-trivial $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ -torsor for the étale topology. As we explain in detail in [13], this perspective makes HT log into an analogue of the Hitchin fibration.

Part 1 of the Theorem makes precise the idea that in order to study étale line bundles on $X \times Y$ where Y is a perfectoid space, it suffices to understand the situation for rigid Y , and vice versa. Part 2 is a geometric upgrade of [14, Theorem 1.3.2] in the proper case and could be described as a statement about “relative p -adic Hodge theory with \mathbb{G}_m -coefficients”.

We already mention some consequences of Theorem 2.7 that we will deduce in the end:

- Corollary 2.9.**
1. $(\mathbf{Pic}_{X, \acute{e}t})^\diamond$ is a v -sheaf on Perf_K .
 2. If we regard $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ as a functor on all of $\mathrm{LSD}_{K, \acute{e}t}$, then it also satisfies the sheaf property for v -covers $Y' \rightarrow Y$ where Y is a smooth rigid space and Y' is perfectoid.
 3. If the rigid Picard functor $\mathbf{Pic}_{X, \acute{e}t}$ is represented by a rigid group G , then its diamondification G^\diamond represents $\mathbf{Pic}_{X, \acute{e}t}^\diamond$.
 4. Conversely, if there is a rigid group G with $G^\diamond \cong \mathbf{Pic}_{X, \acute{e}t}^\diamond$, then G represents $\mathbf{Pic}_{X, \acute{e}t}$.
 5. $\mathbf{Pic}_{X, \acute{e}t}^\diamond$ is represented by a rigid group if and only if $\mathbf{Pic}_{X, v}^\diamond$ is represented by a rigid group.

Remark 2.10. The first part shows in particular that the functor sending $Y \in \mathrm{Perf}_{K, v}$ to the groupoid of analytic line bundles on $X \times Y$ is a v -stack in the sense of [32, §9]. This is similar in spirit to the statement that vector bundles on the Fargues–Fontaine curve satisfy v -descent in the perfectoid variable ([7, Proposition II.2.1] [34, Proposition 19.5.3]).

Remark 2.11. Parts 2–5 might open up new strategies to prove that the rigid Picard functor is always representable by a rigid group whose identity component is semi-abeloid.

In fact, our proof for the étale comparison, i.e. part 1 of Theorem 2.7, works also in higher cohomological degree. More precisely, our proof will give the following stronger statement:

Theorem 2.12. *Let F be one of $\mathcal{O}, \mathcal{O}^\times, \mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{Z}$. Then for $n \geq 0$,*

$$(R^n \pi_{\acute{e}t*} F)^\diamond \xrightarrow{\sim} R^n \pi_{\acute{e}t*}^\diamond F$$

is an isomorphism, and both of these sheaves on $\mathrm{Perf}_{K, \acute{e}t}$ are already v -sheaves.

Remark 2.13. In [15], we show that there is also a higher degree analogue of part 2 of Theorem 2.7: an extension of (3) to a spectral sequence in the category of abelian sheaves on $\mathrm{Perf}_{K, \acute{e}t}$ which is a multiplicative analogue of the Hodge–Tate spectral sequence of X .

2.3. Outline of proof strategy. We now give an outline of the proof of Theorem 2.7, essentially the same line of argument will show Theorem 2.12. The basic strategy is to study step-by-step the following two cohomological diagrams of sheaves on $\text{Perf}_{K,\acute{e}t}$: Following the notation in [14], let us write \mathcal{O}^\times for the sheaf of units on $\text{SmRig}_{K,\acute{e}t}$ represented by \mathbb{G}_m . Let

$$\mathcal{O}_1^\times := 1 + \mathfrak{m}\mathcal{O}^+ \subseteq \mathcal{O}^\times,$$

where we recall that \mathfrak{m} denotes the ideal of topologically nilpotent elements of K^+ . If $K^+ = \mathcal{O}_K$, then \mathcal{O}_1^\times is represented by the open disc of radius 1 around the origin, but in general it is the smaller open subgroup of \mathbb{G}_m given by the union of closed discs of radius < 1 . The role of \mathcal{O}_1^\times is that it is the domain of convergence of the p -adic logarithm $\log : \mathcal{O}_1^\times \rightarrow \mathbb{G}_a$.

Let

$$\overline{\mathcal{O}}^\times := \mathcal{O}^\times / \mathcal{O}_1^\times$$

denote the quotient sheaf. As before, we shall identify these sheaves with their diamondifications, so that we obtain a short exact sequence on $\text{SmRig}_{K,\acute{e}t}$ as well as on $\text{Perf}_{K,\acute{e}t}$

$$0 \rightarrow \mathcal{O}_1^\times \rightarrow \mathcal{O}^\times \rightarrow \overline{\mathcal{O}}^\times \rightarrow 0.$$

Exactly as in [14, Lemma 2.17], we will see that $\overline{\mathcal{O}}^\times$ is in fact already a v -sheaf on Perf_K .

We now apply to the above sequence the two natural transformations

$$(R^n \pi_{\acute{e}t*} -)^\diamond \rightarrow R^n \pi_{\acute{e}t*}^\diamond(-^\diamond) \rightarrow R^n \pi_{v*}^\diamond(-^\diamond).$$

This results in a large commutative diagram of sheaves on $\text{Perf}_{K,\acute{e}t}$

$$\begin{array}{ccccccccc} \pi_{v*}^\diamond \overline{\mathcal{O}}^\times & \longrightarrow & R^1 \pi_{v*}^\diamond \mathcal{O}_1^\times & \longrightarrow & R^1 \pi_{v*}^\diamond \mathcal{O}^\times & \longrightarrow & R^1 \pi_{v*}^\diamond \overline{\mathcal{O}}^\times & \longrightarrow & R^2 \pi_{v*}^\diamond \mathcal{O}_1^\times \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \pi_{\acute{e}t*}^\diamond \overline{\mathcal{O}}^\times & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}_1^\times & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}^\times & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \overline{\mathcal{O}}^\times & \longrightarrow & R^2 \pi_{\acute{e}t*}^\diamond \mathcal{O}_1^\times \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (\pi_{\acute{e}t*} \overline{\mathcal{O}}^\times)^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \mathcal{O}_1^\times)^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \mathcal{O}^\times)^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \overline{\mathcal{O}}^\times)^\diamond & \rightarrow & (R^2 \pi_{\acute{e}t*} \mathcal{O}_1^\times)^\diamond \end{array}$$

in which the bottom two rows are exact with respect to the étale topology and the top row is exact with respect to the v -topology (i.e. we have tacitly applied ν_* to the top row).

We can without loss of generality assume that X is connected. The first step of the proof can then be summarised by saying that we will prove:

Lemma 2.14. *The following hold in the above commutative diagram:*

- (A) *The leftmost horizontal transition maps are 0.*
- (B) *In the fourth column, both maps are isomorphisms.*
- (C) *In the fifth column, the composition of the vertical maps is injective.*

Once this is achieved, it follows formally that the top row is already exact for the étale topology: Indeed, exactness at the second and third term follows from (A) using that ν_* is left-exact, and exactness at the fourth term follows from (B) and (C) by a diagram chase.

At this point, the 5-Lemma (applied once to the bottom maps and once to the compositions) reduces us to proving a variant of the Theorem for \mathcal{O}_1^\times instead of \mathbb{G}_m :

Proposition 2.15.

1. *The map $(R^1 \pi_{\acute{e}t*} \mathcal{O}_1^\times)^\diamond \rightarrow R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}_1^\times$ is an isomorphism.*
2. *There is a short exact sequence of abelian sheaves on $\text{Perf}_{K,\acute{e}t}$*

$$0 \rightarrow R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}_1^\times \rightarrow R^1 \pi_{v*}^\diamond \mathcal{O}_1^\times \xrightarrow{\text{HT log}} H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a \rightarrow 0.$$

In order to prove this, we apply the same strategy as above to the logarithm sequence

$$0 \rightarrow \mu_{p^\infty} \rightarrow \mathcal{O}_1^\times \xrightarrow{\log} \mathcal{O} \rightarrow 0,$$

which results in a commutative diagram of sheaves on $\text{Perf}_{K, \acute{e}t}$

$$(4) \quad \begin{array}{ccccccccc} \pi_{v*}^\diamond \mathcal{O} & \longrightarrow & R^1 \pi_{v*}^\diamond \mu_{p^\infty} & \longrightarrow & R^1 \pi_{v*}^\diamond \mathcal{O}_1^\times & \longrightarrow & R^1 \pi_{v*}^\diamond \mathcal{O} & \longrightarrow & R^2 \pi_{v*}^\diamond \mu_{p^\infty} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \pi_{\acute{e}t*}^\diamond \mathcal{O} & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \mu_{p^\infty} & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O}_1^\times & \longrightarrow & R^1 \pi_{\acute{e}t*}^\diamond \mathcal{O} & \longrightarrow & R^2 \pi_{\acute{e}t*}^\diamond \mu_{p^\infty} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (\pi_{\acute{e}t*} \mathcal{O})^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \mu_{p^\infty})^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \mathcal{O}_1^\times)^\diamond & \rightarrow & (R^1 \pi_{\acute{e}t*} \mathcal{O})^\diamond & \rightarrow & (R^2 \pi_{\acute{e}t*} \mu_{p^\infty})^\diamond \end{array}$$

in which again the bottom two rows are exact with respect to the étale topology and the top row is exact with respect to the v -topology. The fourth row of this diagram is described by Proposition 2.6. We use this description to prove:

Lemma 2.16. *The following hold in the above commutative diagram:*

- (D) *The leftmost horizontal transition maps are 0.*
- (E) *In the second and fifth column, all maps are isomorphisms.*
- (F) *Further towards the right, the map $(R^2 \pi_{\acute{e}t*} \mathcal{O})^\diamond \rightarrow (R^2 \pi_{v*} \mathcal{O})^\diamond$ is injective.*

Using parts (D), (E), and a direct argument describing the vertical cokernels in the middle of the top two rows, we see that all sheaves in the above diagram are in fact v -sheaves, so that may regard (4) as a commutative diagram of v -sheaves with exact rows. Part (F) then implies part (C) above. Parts (D) and (E) will be enough to prove part 1 of Proposition 2.15.1, and left-exactness in Proposition 2.15.2. Finally, we will use a relative version of the argument in [14, §3.5] to prove the right-exactness, using the diamantine universal cover. This will complete the proof of the Theorem.

From all of the above steps, the key step is arguably the proof of (B): This is where we need to study the transition from functors on smooth rigid spaces to functors on perfectoid spaces in great detail. We do this by proving a very general rigid approximation lemma. This will also be handy to complete some of the other steps, although only (B) uses it in its full force. Proving the rigid approximation lemma is the goal of the next section.

3. A RIGID APPROXIMATION LEMMA

We now begin the proof of Theorem 2.7 with a rigid approximation lemma for the sheaf $\overline{\mathcal{O}}^\times = \mathcal{O}^\times / \mathcal{O}_1^\times$ introduced in Section 2.3. This will be required for step (B). For the statement we use tilde-limits [20, (2.4.1)][33, §2.4] as well as a slight strengthening for affinoids:

Definition 3.1. For a cofiltered inverse system of adic spaces $(X_i)_{i \in I}$ with qcqs transition maps, and an adic space X_∞ with compatible maps $X_\infty \rightarrow X_i$ for all $i \in I$, we write

$$X_\infty \sim \varprojlim_{i \in I} X_i$$

if on the underlying topological spaces, the maps induce a homeomorphism $|X_\infty| = \varprojlim |X_i|$, and if there is a cover of X_∞ by affinoid opens U_∞ for which the map $\varprojlim_U \mathcal{O}(U) \rightarrow \mathcal{O}(U_\infty)$ has dense image, where $U \subseteq X_i$ runs through all affinoid opens through which $U_\infty \rightarrow X_i$ factors, and all i . If moreover all X_i and X_∞ are affinoid, we write

$$X_\infty \approx \varprojlim_{i \in I} X_i$$

if already the global sections $\varprojlim \mathcal{O}(X_i) \rightarrow \mathcal{O}(X_\infty)$ have dense image.

Proposition 3.2. *Let Y be an affinoid perfectoid space over K and let $(Y_i)_{i \in I}$ be a cofiltered inverse system of affinoid smooth rigid spaces such that $Y \approx \varprojlim Y_i$. Let X be a qcqs adic space over K that is either smooth or perfectoid. Let $U_i \rightarrow X \times Y_i$ be a qcqs étale map and set $U_j := U_i \times_{Y_i} Y_j$ and $U := U_i \times_{Y_i} Y$. Then for all $n \geq 0$, the natural map*

$$\varinjlim_{j \geq i} H_{\text{ét}}^n(U_j, F) \rightarrow H_{\text{ét}}^n(U, F)$$

is an isomorphism for $F = \overline{\mathcal{O}}^\times$ or $F = \mathcal{O}^+/\varpi$ for any $0 \neq \varpi \in \mathfrak{m}$.

Remark 3.3. • A priori, the fibre product $X \times Y_i$ is in the category of diamonds over $\text{Spd}(K, K^+)$. But since Y_i is smooth, this is represented by a sousperfectoid adic space in the sense of Hansen–Kedlaya [34, §6.3]: the fibre product of X and Y_i over $\text{Spa}(K)$ in the category of uniform adic spaces.

- The analogue of the proposition for $F = \mathcal{O}^+$ and \mathcal{O}^\times fails already for $n = 0$.
- With some more work, the assumption of the Proposition can be weakened, e.g. it also holds in characteristic p : The proof for perfectoid X works without changes. For rigid X , one can then use local sections of Frobenius to descend from the perfection.

Proof. The proof will be completed by a series of lemmas. We start with an easy observation:

Lemma 3.4. *In the situation of Proposition 3.2, we have:*

1. $U = \varprojlim_{j \geq i} U_j$ as diamonds.
2. If $U \rightarrow X \times Y$ is an étale cover, then so is $U_j \rightarrow X \times Y_j$ for $j \gg i$.

Proof. We have $Y^\diamond = \varprojlim_{i \in I} Y_i^\diamond$ by [33, Proposition 2.4.5]. Part (i) follows since limits commute with fibre product.

Part 2 follows from 1 due to the qcqs assumption: Namely, since $U_j \rightarrow X \times Y_j$ is étale, it is open [20, Proposition 1.7.8], so we can without loss of generality replace U_j by its quasi-compact open image. The statement now follows from the following Lemma. \square

Lemma 3.5 ([29, Lemma 6.13.(iv)]). *Let $T = \varprojlim T_i$ be a cofiltered inverse limit of spectral spaces with spectral transition maps. Let $U \subseteq T$ be a quasi-compact open such that $T \rightarrow T_i$ factors through U . Then already some $T_j \rightarrow T_i$ factors through U .*

Proof. For any $q : T_j \rightarrow T_i$ in the inverse system, set $Z_j := T_j \setminus q^{-1}(U)$. Then the assumptions imply $\varprojlim Z_j = \emptyset$. The desired result now follows from [5, 0A2W]. \square

In order to continue, we need in the following several subcategories of the étale site:

Definition 3.6. 1. Let Z be a locally spatial diamond over K . Let $Z_{\text{ét-qcqs}} \subseteq Z_{\text{ét}}$ be the full subcategory of quasi-compact quasi-separated étale morphisms $U \rightarrow Z$. For any adic space Z , this also defines $Z_{\text{ét-qcqs}}$ via the identification $Z_{\text{ét}} = Z_{\text{ét}}^\diamond$.

2. If Z is an affinoid adic space over K , let $Z_{\text{stdét}} \subseteq Z_{\text{ét}}$ be the full subcategory of objects $Z' \rightarrow Z$ which are successive compositions of rational open immersions and finite étale maps. We call such maps standard-étale. By [32, Lemmas 11.31 and 15.6], these form a basis of $Z_{\text{ét}}$. Note that we have $Z_{\text{stdét}} \subseteq Z_{\text{ét-qcqs}} \subseteq Z_{\text{ét}}$.

Next, we explain that in order to prove Proposition 3.2 for all $n \geq 0$, we can reduce to the case of $n = 0$. We first note that for $n = 0$, the statement is the following:

Claim 3.7. *In the situation of Proposition 3.2, for any $U_i \in (X \times Y_i)_{\text{ét-qcqs}}$ with pullbacks $U_j \rightarrow X \times Y_j$ and $U \rightarrow X \times Y$, we have*

$$\overline{\mathcal{O}}^\times(U) = \varinjlim_{j \geq i} \overline{\mathcal{O}}^\times(U_j), \quad \mathcal{O}^+/\varpi(U) = \varinjlim_{j \geq i} \mathcal{O}^+/\varpi(U_j).$$

Suppose that Claim 3.7 holds true. Then the case of $n \geq 0$ follows from very general results on cohomology in inverse limit topoi:

Lemma 3.8. *Let $Z = \varprojlim_{i \in I} Z_i$ be a cofiltered inverse limit of spatial diamonds over $\mathrm{Spa}(K, K^+)$. Let F be an abelian sheaf on $\mathrm{LSD}_{K, \acute{e}t}$. Assume that for all $i \in I$ and $U_i \in Z_{i, \acute{e}t\text{-qcqs}}$ with pullbacks $U_j = U_i \times_{Z_i} Z_j$ and $U = U_i \times_{Z_i} Z_\infty$ we have*

$$F(U) = \varinjlim_{j \geq i} F(U_j).$$

Then for all $n \geq 0$,

$$H^n(U, F) = \varinjlim_{j \geq i} H^n(U_j, F).$$

Proof. Since the Z_i are spatial, we have by [32, Proposition 11.23] an equivalence of sites $Z_{\acute{e}t\text{-qcqs}} = 2\text{-}\varinjlim_{i \in I} Z_{i, \acute{e}t\text{-qcqs}}$. Write $\mu_j : Z \rightarrow Z_j$ for the natural projection, then by [5, 09YN] our assumptions imply that $F_Z = \varinjlim_{j \geq i} \mu_j^{-1} F_{Z_j}$, from which the statement follows formally by [35, VI Théorème 8.7.3] or [5, 09YP]. In fact, we will later only need the case $n = 1$, in which case this is a simple Čech argument. \square

We have thus reduced Proposition 3.2 to Claim 3.7. We now first prove Claim 3.7 for qcqs perfectoid X . In this case, we can further reduce the claim to the following statement:

Claim 3.9. *In the situation of Proposition 3.2, assume further that X is affinoid perfectoid. Then for any $U_i \in (X \times Y_i)_{\mathrm{std\acute{e}t}}$ with pullbacks $U_j \in (X \times Y_j)_{\mathrm{std\acute{e}t}}$ for $j \geq i$ and $U \in (X \times Y)_{\mathrm{std\acute{e}t}}$, we have*

$$\mathcal{O}^\times(U)/\mathcal{O}_1^\times(U) = \varinjlim_{j \geq i} \mathcal{O}^\times(U_j)/\mathcal{O}_1^\times(U_j), \quad \mathcal{O}^+(U)/\varpi = \varinjlim_{j \geq i} \mathcal{O}^+(U_j)/\varpi.$$

Indeed, suppose we know Claim 3.9. Then using the equivalence of sites

$$(X \times Y)_{\mathrm{std\acute{e}t}} = 2\text{-}\varinjlim_{i \in I} (X \times Y_i)_{\mathrm{std\acute{e}t}}$$

and the fact that $(X \times Y)_{\mathrm{std\acute{e}t}}$ is a basis for $(X \times Y)_{\acute{e}t}$, it follows upon sheafification that

$$\overline{\mathcal{O}}^\times(U) = \varinjlim_{j \geq i} \overline{\mathcal{O}}^\times(U_j), \quad \mathcal{O}^+/\varpi(U) = \varinjlim_{j \geq i} \mathcal{O}^+/\varpi(U_j).$$

More generally, we now also obtain these last two equalities if $U_i \in (X \times Y_i)_{\acute{e}t\text{-qcqs}}$, because any such U_i can be covered by finitely many objects of $(X \times Y_i)_{\mathrm{std\acute{e}t}}$ and their intersections are again covered by finitely many objects of $(X \times Y_i)_{\mathrm{std\acute{e}t}}$ due the qcqs assumption.

In a second step, this now implies Claim 3.7 for qcqs perfectoid X , by applying the same covering argument to a finite cover of X by affinoid perfectoid subspaces.

We now prove Claim 3.9 step by step. We first treat the case that $X = \mathrm{Spa}(K, K^+)$ where (K, K^+) is a perfectoid field. In fact, for the following discussion until Lemma 3.13 inclusively, we can allow the greater generality that (K, K^+) is any non-archimedean field of residue characteristic p . We fix a pseudo-uniformiser $0 \neq \varpi \in \mathfrak{m}$.

Lemma 3.10. *Let $(Y_i)_{i \in I}$ be a cofiltered inverse system of affinoid adic spaces over K with an affinoid tilde-limit $Y \approx \varprojlim Y_i$. Then the following maps are isomorphisms:*

$$\varinjlim_{i \in I} \mathcal{O}^\times(Y_i)/\mathcal{O}_1^\times(Y_i) \rightarrow \mathcal{O}^\times(Y)/\mathcal{O}_1^\times(Y), \quad \varinjlim_{i \in I} \mathcal{O}^+(Y_i)/\varpi \rightarrow \mathcal{O}^+(Y)/\varpi.$$

Proof. Let $f \in \mathcal{O}^+(Y)$. Then we approximate f by some $f_i \in \mathcal{O}(Y_i)$ whose image in $\mathcal{O}(Y)$ satisfies $|f_i - f| \leq |\varpi|$. In particular, we then have $f_i \in \mathcal{O}^+(Y)$. The condition $|f_i| \leq 1$ defines a quasi-compact open subspace U of Y_i through which $Y \rightarrow Y_i$ factors. We may apply Lemma 3.5 to this situation since any morphism between affinoid analytic adic spaces

is spectral. Consequently, there is $j \geq i$ such that $Y_j \rightarrow Y_i$ factors through U , which means that the image f_j of f_i in $\mathcal{O}(Y_j)$ is already in $\mathcal{O}^+(Y_j)$. This shows surjectivity.

Injectivity follows by a similar argument: If $f_i \in \mathcal{O}^+(Y_i)$ is such that its image $f \in \mathcal{O}^+(Y)$ is already in $\varpi\mathcal{O}^+(Y)$, then some $Y_j \rightarrow Y_i$ factors through the quasi-compact open defined by $|f_i| \leq |\varpi|$ because $Y \rightarrow Y_i$ does. Thus f_i goes to $0 \in \mathcal{O}^+(Y_j)/\varpi$.

The proof for \mathcal{O}^\times is similar as the proof of [14, Lemma 2.17] goes through verbatim: To see injectivity, let $g_i \in \mathcal{O}_1^\times(Y_i)$ be in the kernel. Then since Y is quasi-compact, there is $\epsilon > 0$ such that $|g_i - 1| \leq |\varpi|^\epsilon$ on Y . By Lemma 3.5 we have $g_i \in \mathcal{O}_1^\times(Y_j)$ for some $j \gg i$.

To see that the map is surjective, let $f \in \mathcal{O}^\times(Y)$. By assumption, we can find approximating sequences $f_i \rightarrow f$ and $f'_i \rightarrow f^{-1}$ with $f_i, f'_i \in \mathcal{O}(Y_i)$. Then $f_i f'_i \rightarrow 1$ and thus $f_i f'_i \in \mathcal{O}_1^\times(Y)$ for $i \gg 0$. By the above argument, it follows that $f_i f'_i \in \mathcal{O}_1^\times(Y_j)$ for some $j \geq i$. But then $f_i \in \mathcal{O}^\times(Y_j)$, which implies that f_i is in the image of the map. \square

According to Lemma 3.10, in order to prove Claim 3.9, it suffices to prove that $U \approx \varprojlim U_i$. We now first prove this when $U_i = X \times Y_i$. Recall that X is affinoid perfectoid in Claim 3.9.

Lemma 3.11. *Let X and Y be affinoid perfectoid spaces and assume we have a tilde-limit $Y \approx \varprojlim Y_i$ for some smooth affinoid rigid spaces Y_i over K . Then $X \times Y \approx \varprojlim X \times Y_i$.*

Proof. Since X and Y are affinoid perfectoid, we have

$$\mathcal{O}^+(X \times Y)/\varpi \stackrel{a}{=} \mathcal{O}^+(X) \otimes_{K^+} \mathcal{O}^+(Y)/\varpi.$$

The diamond $X \times Y_i$ is represented by an affinoid adic space (cf Remark 3.3): This is defined by the Huber pair $(B_i[1/\varpi], B_i)$ given by setting $A_i := \mathcal{O}^+(X) \hat{\otimes} \mathcal{O}^+(Y_i)$ and defining B_i to be the integral closure of the image of A_i in $A_i[1/\varpi]$. Consider now the composition

$$\varprojlim A_i \rightarrow \varprojlim B_i \rightarrow \mathcal{O}^+(X \times Y) \stackrel{a}{=} \mathcal{O}^+(X) \hat{\otimes}_{K^+} \mathcal{O}^+(Y).$$

Here the second map is \mathcal{O}^+ evaluated on $X \times Y \rightarrow X \times Y_i$. We wish to see that this second map is an almost isomorphism mod ϖ . This will imply that $X \times Y \approx \varprojlim X \times Y_i$.

By Lemma 3.10, the assumptions imply that $\mathcal{O}^+(Y)/\varpi = \varprojlim \mathcal{O}^+(Y_i)/\varpi$, hence the above composition is an almost isomorphism mod ϖ . The statement now follows formally: Let us axiomatise the argument for later reference.

Lemma 3.12. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms of \mathcal{O}_K^a -modules such that*

- B is ϖ -torsionfree
- $f[1/\varpi]$ and $(g \circ f)/\varpi$ are both isomorphisms.

Then g/ϖ is an isomorphism.

Proof. Let $T \subseteq A$ be the ϖ -power torsion submodule, then $T/\varpi \hookrightarrow A/\varpi$ is injective. The map f/ϖ is almost injective since $(g \circ f)/\varpi$ is. On the other hand, $T \rightarrow B$ is trivial since B is ϖ -torsionfree. This implies that $T/\varpi = 0$. In particular, we have $(A/T)/\varpi = A/\varpi$. We also have $(A/T)[1/\varpi] = A[1/\varpi]$. We may thus replace A by A/T and assume without loss of generality that A is ϖ -torsionfree. In particular, f is then injective since $f[1/\varpi]$ is.

In this situation, the cokernel of f is ϖ -power torsion (since $f[1/\varpi]$ is an isomorphism) but also ϖ -torsionfree (since f/ϖ is injective). Consequently, f is an isomorphism, thus f/ϖ is an isomorphism, and hence so is g/ϖ given that $(g \circ f)/\varpi$ is. \square

Applying Lemma 3.12 to the given sequence finishes the proof of Lemma 3.11. \square

Finally, in order to prove Claim 3.9, it remains to add a standard-étale map on top of $X \times Y_i$. Setting $Z := X \times Y$ and $Z_i := X \times Y_i$ to simplify notation, we wish to see:

Lemma 3.13. *Let $Z \approx \varprojlim Z_i$ be an affinoid perfectoid tilde-limit of affinoid adic spaces Z_i over K . Let $U_i \rightarrow Z_i$ be an object of $Z_{i, \text{stdét}}$. For $j \geq i$ write $U_j := U_i \times_{Z_i} Z_j$ and $U := U_i \times_{Z_i} Z$. If all of these are adic spaces, then $U \approx \varprojlim_{j \geq i} U_j$.*

Proof. We can prove this separately in the cases of finite étale maps and rational localisation. Case 1: $U_i \rightarrow Z_i$ finite étale. Write $S_j = \mathcal{O}(Z_j)$ and $S := \mathcal{O}(Z)$. Similarly, for any $j \geq i$, let $R_j := \mathcal{O}(U_j)$ and $R := \mathcal{O}(U)$. By [32, Lemma 15.6] and [21, Lemma 8.2.17.(i)], the map $S_i \rightarrow R_i$ is finite étale and we have $R_j = R_i \otimes_{S_i} S_j$ and $R = R_i \otimes_{S_i} S$. To see that $\varinjlim_{j \geq i} R_j \rightarrow R$ has dense image, it now suffices to see that we can approximate simple tensors $r \otimes s$ in $R_i \otimes_{S_i} S = R$. For this we use that $\varinjlim \mathcal{O}(S_j) \rightarrow \mathcal{O}(S)$ has dense image to find a sequence $s_j \in S_j$ with $s_j \rightarrow s$. This implies $r \otimes s_j \rightarrow r \otimes s$, showing the desired dense image property. By [20, Remark 2.4.3.(ii)], we also have $|U| = \varprojlim |U_j|$. Thus $U \approx \varprojlim U_j$. Case 2: $U_i \rightarrow Z_i$ a rational open immersion. Since Z is affinoid perfectoid, we have $S^+ := \mathcal{O}^+(U) \stackrel{a}{=} \mathcal{O}^\circ(U) = S^\circ$. It therefore suffices to prove that for all $n \in \mathbb{N}$,

$$\varinjlim \mathcal{O}^+(U_j)/\varpi^n \rightarrow \mathcal{O}^\circ(U)/\varpi^n$$

is an almost isomorphism. Second, by [29, Lemma 6.4], the rational open $U \subseteq Z$ is of the form $Z(f_1, \dots, f_n/g)$ for some $f_1, \dots, f_n, g \in \sharp(\mathcal{O}^{b^+}(Z))$ with $f_n = \varpi^N$ for some N , and can be written as

$$U = \text{Spa}(R, R^+) \quad \text{where} \quad R^+ \stackrel{a}{=} S^\circ \langle (f_1/g)^{1/p^\infty}, \dots, (f_n/g)^{1/p^\infty} \rangle.$$

Set $\epsilon := |\varpi^N|$. Then for every $l \in \mathbb{N}$, we can find j_l large enough such that there are $f_{1,l}, \dots, f_{n,l}, g_l$ in $\mathcal{O}(Z_{j_l})$ such that on Z we have for all $i = 1, \dots, n$:

$$|f_{i,l} - f_i^{1/p^l}| \leq \epsilon \quad \text{and} \quad |g_l - g^{1/p^l}| \leq \epsilon.$$

Then on Z , the conditions $|f_{i,l}| \leq |g_l|$ and $|f_i^{1/p^l}| \leq |g^{1/p^l}|$ are equivalent, and thus cut out the same rational open subspace of Z . For any $j \geq j_l$, let

$$U_j := Z_j(f_{1,l}, \dots, f_{n,l}/g_l),$$

then this means that $U = Z \times_{Z_j} U_j$. Moreover, we have a natural isomorphism

$$S^\circ \langle (f_i/g)^{1/p^l} \rangle / \varpi = S^\circ \langle f_{i,l}/g_l \rangle / \varpi$$

given explicitly by

$$\begin{aligned} S^\circ / \varpi [T_1, \dots, T_n] / (T_i g^{1/p^l} - f_i^{1/p^l}) &\xrightarrow{\sim} S^\circ / \varpi [T_1, \dots, T_n] / (T_i g_l - f_{i,l}), \\ T_i &\mapsto \left(1 + \frac{g^{1/p^l} - g_l}{g_l}\right)^{-1} \left(T_i - \frac{f_{i,l} - f_i^{1/p^l}}{g_l}\right). \end{aligned}$$

Under these compatible identifications, in the limit over l , it makes sense to write

$$R^\circ / \varpi \stackrel{a}{=} S^\circ / \varpi \left[\frac{f_{i,l}}{g_l} \mid i = 1, \dots, n \text{ and } l \in \mathbb{N} \right].$$

For fixed l , Lemma 3.5 implies that for $j \gg j_l$ we have $f_{i,l}, g_l \in \mathcal{O}^+(Z_j)$. Let

$$A_{j,l} := \mathcal{O}^+(Z_j) \langle f_{i,l}/g_l \mid i = 1, \dots, n \rangle, \quad B_j := \mathcal{O}^+(U_j).$$

Explicitly, B_j is the integral closure of the image of $A_{j,l}$ in $A_{j,l}[\frac{1}{\varpi}]$. In particular, we have

$$B_j[1/\varpi] = A_{j,l}[1/\varpi].$$

We now observe that by construction, for any fixed l , the map

$$\varinjlim_{j \geq j_l} A_{j,l} / \varpi = \varinjlim_{j \geq j_l} (\mathcal{O}^+(Z_j) / \varpi) [f_{i,l}/g_l] \rightarrow (S^\circ / \varpi) [f_{i,l}/g_l] \stackrel{a}{=} (S^\circ / \varpi) [(f_i/g)^{1/p^l}]$$

is an almost isomorphism since this was true before tensoring with $\mathcal{O}^+(Z_{j_l})[f_{i,l}/g_l]$. Taking the colimit over l , this shows that also

$$\varinjlim_{l \in \mathbb{N}} \varinjlim_{j \geq j_l} A_{j,l} / \varpi \stackrel{a}{=} (S^\circ / \varpi) [(f_i/g)^{1/p^\infty}] \stackrel{a}{=} \mathcal{O}^+(U) / \varpi$$

is an almost isomorphism. The desired statement now follows from Lemma 3.12 applied to the sequence

$$\varinjlim_l \varinjlim_{j \geq j_l} A_{i,l} \rightarrow \varinjlim_j B_j \rightarrow \mathcal{O}^+(U)/\varpi.$$

This finishes the proof of Lemma 3.13. \square

Combining Lemma 3.13 with Lemma 3.10, we have thus proved Claim 3.9, which finishes the proof of Proposition 3.2 for perfectoid X .

In order to deduce the case of smooth rigid X , it again suffices to prove Claim 3.7 in this case. To this end, we return to our original setup that K is a perfectoid field over \mathbb{Q}_p . We then have the following consequence of the perfectoid case:

Lemma 3.14. *Let X be a smooth rigid space and let Y be affinoid perfectoid over K . Then on $X \times Y$, we have*

$$\mathcal{O}_{\text{ét}}^+/\varpi \stackrel{a}{=} \nu_*(\mathcal{O}_v^+/\varpi).$$

In particular, this holds on any smooth rigid space. Similarly, $\overline{\mathcal{O}}_{\text{ét}}^\times = \nu_ \overline{\mathcal{O}}_v^\times$.*

Remark 3.15. We show a more general statement in [12, Proposition 2.14], which also says that in fact, we can write $=$ instead of $\stackrel{a}{=}$. But Lemma 3.14 is much easier to see:

Proof. The statement is local on X . We may therefore assume that X is affinoid and that we can find an affinoid perfectoid pro-finite-étale Galois cover $\tilde{X} = \varprojlim X_i \rightarrow X$ with group $G = \varprojlim G_i$. Let $U \in (X \times Y)_{\text{stdét}}$ be standard-étale, in particular affinoid, and let $\tilde{U} \rightarrow \tilde{X} \times Y$ be the pullback. We can without loss of generality assume that $\tilde{U} = \varprojlim U_i \rightarrow U$ is affinoid perfectoid and pro-finite-étale Galois with group G . Then we have $(\mathcal{O}_{\text{ét}}^+/\varpi)(\tilde{U}) \stackrel{a}{=} (\mathcal{O}_v^+/\varpi)(\tilde{U})$ since \tilde{U} is perfectoid and \mathcal{O}^+ is almost acyclic on affinoid perfectoids for both the étale and the v -topology by [29, Proposition 7.13] and [32, Proposition 8.8], respectively. Consequently,

$$(\mathcal{O}_v^+/\varpi)(U) = (\mathcal{O}_v^+/\varpi)(\tilde{U})^G \stackrel{a}{=} (\mathcal{O}_{\text{ét}}^+/\varpi)(\tilde{U})^G = \varinjlim_i (\mathcal{O}_{\text{ét}}^+/\varpi)(U_i)^{G_i} = \mathcal{O}_{\text{ét}}^+/\varpi(U),$$

where the third step follows from Lemma 3.13 and Lemma 3.10 upon étale sheafification.

The case of $\overline{\mathcal{O}}^\times$ is analogous once we know that

$$\overline{\mathcal{O}}_{\text{ét}}^\times(\tilde{U}) = \overline{\mathcal{O}}_v^\times(\tilde{U}).$$

To see this, let τ be either the étale or the v -topology, then we have the exponential sequence from [14, Lemma 2.18]:

$$0 \rightarrow \mathcal{O}_\tau \xrightarrow{\text{exp}} \varinjlim_{x \rightarrow x^p} \mathcal{O}_\tau^\times \rightarrow \overline{\mathcal{O}}_\tau^\times \rightarrow 1.$$

This remains short exact after evaluating at \tilde{U} since \mathcal{O} is acyclic on affinoid perfectoids in both topologies. The desired statement now follows from the fact that $\mathcal{O}_{\text{ét}}(\tilde{U}) = \mathcal{O}_v(\tilde{U})$ and $\mathcal{O}_{\text{ét}}^\times(\tilde{U}) = \mathcal{O}_v^\times(\tilde{U})$ since \tilde{U} is perfectoid. \square

It follows that in order to prove Claim 3.7 for rigid X , it suffices to prove the statement for $\mathcal{O}_{\text{ét}}^+/\varpi$ replaced by \mathcal{O}_v^+/ϖ , and $\overline{\mathcal{O}}_{\text{ét}}^\times$ replaced by $\overline{\mathcal{O}}_v^\times$. But since X is a qcqs smooth rigid space, there is a v -cover of X by a qcqs perfectoid space \tilde{X} such that $\tilde{X} \times_X \tilde{X}$ is qcqs perfectoid. Therefore the result now follows from the statement for perfectoid X .

This finishes the proof of Proposition 3.2. \square

Our main application of Proposition 3.2 is that it implies the first part of Lemma 2.14.(B):

Corollary 3.16. *Let $\pi : X \rightarrow \mathrm{Spa}(K)$ be a qcqs smooth rigid space. Then the morphism of sheaves on $\mathrm{Perf}_{K,\acute{\mathrm{e}}\mathrm{t}}$*

$$(R^n \pi_{\acute{\mathrm{e}}\mathrm{t}*} \overline{\mathcal{O}}^\times)^\diamond \xrightarrow{\sim} R^n \pi_{\acute{\mathrm{e}}\mathrm{t}*}^\diamond \overline{\mathcal{O}}^\times$$

from (1) is an isomorphism for all $n \geq 0$. Similarly for \mathcal{O}^+/ϖ for any $0 \neq \varpi \in K^+$.

Proof. Unravelling the definition, we see that both sides are the étale sheafifications of the presheaves on Perf_K defined as follows: The left hand side is

$$Y \mapsto \varinjlim_{Y \rightarrow Z} H_{\acute{\mathrm{e}}\mathrm{t}}^n(X \times Z, \overline{\mathcal{O}}^\times)$$

where $Y \rightarrow Z$ ranges through all morphisms to affinoid smooth rigid spaces Z over K , and where we use $Y_{\acute{\mathrm{e}}\mathrm{t}\text{-qcqs}} = 2\text{-}\varinjlim Z_{\acute{\mathrm{e}}\mathrm{t}\text{-qcqs}}$ to see that it suffices to sheafify with respect to Y .

The right hand side is

$$Y \mapsto H_{\acute{\mathrm{e}}\mathrm{t}}^n(X \times Y, \overline{\mathcal{O}}^\times).$$

We would like to argue that these two presheaves are isomorphic on affinoid perfectoid Y by Proposition 3.2 in the case of $U = X \times Y$. To see that this applies, it remains to prove:

Proposition 3.17. *Let (K, K^+) be a non-archimedean field over \mathbb{Z}_p and let $Y \rightarrow Y_0$ be a morphism of affinoid adic spaces over (K, K^+) . Assume that Y is uniform and that for any affinoid smooth morphism $Y_1 \rightarrow Y_0$ of topologically finite type also Y_1 is uniform. For example, this is satisfied when Y and Y_0 are sousperfectoid. Let $(Y \rightarrow Y_i)_{i \in I}$ be the cofiltered inverse system of all morphisms of adic spaces from Y into affinoid adic spaces Y_i that are smooth of topologically finite type over Y_0 . Then*

$$Y \approx \varprojlim_{Y \rightarrow Y_i} Y_i.$$

In particular, if K is perfectoid, then for any abelian sheaf F on $\mathrm{SmRig}_{K,\acute{\mathrm{e}}\mathrm{t}}$ and any $n \geq 0$, we have

$$H_{\acute{\mathrm{e}}\mathrm{t}}^n(Y, F^\diamond) = \varinjlim_i H_{\acute{\mathrm{e}}\mathrm{t}}^n(Y_i, F).$$

All of this remains true if we instead take the Y_i to be open subspaces of unit balls over Y_0 .

Proof. That the assumptions are satisfied when Y and Y_0 are sousperfectoid follows from [34, Propositions 6.3.3 and 6.3.4].

Let $0 \neq \varpi \in K^+$ be a pseudo-uniformiser. As the very first step, we consider a different inverse system that is not yet smooth: Write $Y = \mathrm{Spa}(S, S^+)$ and $Y_0 = \mathrm{Spa}(S_0, S_0^+)$ and let \mathcal{J} be the partially ordered set of finite subsets of S^+ . For $J \in \mathcal{J}$, let S_J be the image of

$$\phi_J : S_0 \langle X_j | j \in J \rangle \rightarrow S, \quad X_j \mapsto j,$$

and let S_J^+ be the integral closure of the image $S_{J,0}$ of $S_0^+ \langle X_j | j \in J \rangle$ in S_J . Then $S_J \subseteq S$, and the $Z_J := \mathrm{Spa}(S_J, S_J^+)$ form a cofiltered inverse system of adic spaces of topologically finite type over Y_0 such that $\varinjlim_{J \in \mathcal{J}} S_J^+ \rightarrow S^+$ is an isomorphism by construction: In fact, already $\varinjlim_{J \in \mathcal{J}} S_{J,0} \rightarrow S^+$ is an isomorphism. We thus have $Y \approx \varprojlim_{J \in \mathcal{J}} Z_J$ where we use [33, Proposition 2.4.2] to see the required statement about the underlying topological spaces. Here we use that S is uniform, so S^+ has the ϖ -adic topology.

Passing to the inverse system I in the Lemma, we note that any morphism $Y \rightarrow Y_i$ to a smooth affinoid adic space over Y_0 factors through some Z_J . We thus get a well-defined map

$$(5) \quad \varinjlim_{i \in I} \mathcal{O}^+(Y_i) \rightarrow \varinjlim_{J \in \mathcal{J}} \mathcal{O}^+(Z_J) = \varinjlim_{J \in \mathcal{J}} S_{J,0}$$

and it suffices to prove that this becomes an isomorphism after ϖ -adic completion.

We first note that the map is surjective: this is because any Z_J has by its definition via ϕ_J a closed immersion into a closed ball $Z_J \hookrightarrow \mathbb{B}^J$, and \mathbb{B}^J is smooth and thus appears as one of the Y_i on the left hand side. Thus $S_{J,0}$ is in the image.

To see that it is in fact an isomorphism after ϖ -adic completion, let $Y \rightarrow Z = \mathrm{Spa}(R, R^+)$ be any morphism into a smooth affinoid over Y_0 . Let $Z_0 \subseteq Z$ be the closure of the image, i.e. the closed subspace cut out by the kernel $N \subseteq R$ of the corresponding map $R \rightarrow S$.

Claim 3.18. *We have $Z_0 \approx \varprojlim_{Z_0 \subseteq U \subseteq Z} U$ where U ranges through the rational open neighbourhoods of Z_0 in Z .*

Proof. Clearly $\varinjlim \mathcal{O}(U) \rightarrow \mathcal{O}(Z_0)$ is even surjective, so it suffices to check the condition on topological spaces: Both sides are subspaces of $|Z|$, so the map is necessarily a homeomorphism onto its image. It is also surjective: Let $x \in |Z| \setminus |Z_0|$, then there is $f \in N$ such that $|f(x)| \neq 0$. Since ϖ is topologically nilpotent and Z is quasi-compact, it follows that there is k such that $|f(x)| > |\varpi^k|$ on Z . Thus $|f| \leq |\varpi^k|$ defines a rational open neighbourhood of Z_0 that does not contain x . \square

Lemma 3.10 now implies that $\varinjlim_{Z_0 \subseteq U \subseteq Z} \mathcal{O}^+(U) \rightarrow \mathcal{O}^+(Z_0)$ becomes an isomorphism mod ϖ^k . It follows that both sides of (5) agree mod ϖ^k with

$$\varinjlim_{Y \rightarrow U \subseteq \mathbb{B}_{Y_0}^n} \mathcal{O}^+(U)/\varpi^k,$$

where the index category consists of morphisms from Y into rational open subspaces $U \subseteq \mathbb{B}_{Y_0}^n$ of rigid polydiscs. This proves the first part of Lemma 3.17.

The part about cohomology now follows from Lemma 3.8. Alternatively, we could follow the argument in [32, Proposition 14.9], or in [30, Lemma 3.16, Corollary 3.17]. \square

Proposition 3.17 applied to $Y_0 = \mathrm{Spa}(K)$ thus finishes the proof of Corollary 3.16. \square

A similar but much easier argument as in Corollary 3.16 completes step (E) of Lemma 2.16:

Corollary 3.19. *For any $n, N \in \mathbb{N}$, the map from (1) for $\mathcal{F} = \mathbb{Z}/N\mathbb{Z}$ is an isomorphism*

$$(R^n \pi_{\acute{e}t*} \mathbb{Z}/N\mathbb{Z})^\diamond \xrightarrow{\sim} R^n \pi_{\acute{e}t*}^\diamond \mathbb{Z}/N\mathbb{Z}.$$

Proof. Arguing as in the last proof, we see that the first term is the sheafification of

$$Y \mapsto \varinjlim_{Y \rightarrow Z} H_{\acute{e}t}^n(X \times Z, \mathbb{Z}/N),$$

whereas the second term is the sheafification of $Y \mapsto H_{\acute{e}t}^n(X \times Y, \mathbb{Z}/N)$. The two presheaves agree by [32, Proposition 14.9] which applies by Proposition 3.17. \square

In summary, we have in this section completed those parts of Lemma 2.14.(B) and Lemma 2.16.(E) that concern the comparison of the bottom two rows.

4. COHOMOLOGY OF PRODUCTS OF RIGID WITH PERFECTOID SPACES

The main aim of this section is to complete the proofs of Lemma 2.14 and Lemma 2.16. We start with cohomological computations for the sheaf \mathcal{O} , for which we also prove the results described in the first part of Section 2.2, namely Proposition 2.4 and Proposition 2.6.

4.1. Cohomology of \mathcal{O} . We begin with a few general lemmas on the cohomology of “mixed tensor products” of rigid with perfectoid spaces, which are essentially coherent base-change results. For these we crucially use Kiehl’s Theorem that $R\Gamma(X, \mathcal{O})$ is perfect for any proper rigid space X .

Lemma 4.1. *Let X be an affinoid rigid space over K and let Y be an affinoid perfectoid space. Then for $n > 0$, we have $H_{\text{ét}}^n(X \times Y, \mathcal{O}) = 0$.*

Proof. This is true in much greater generality by an application of [21, Theorem 8.2.22(c)]. This applies here because étale maps that factor into rational embeddings and finite étale maps form a basis for $(X \times Y)_{\text{ét}}$ by Proposition [32, Proposition 11.31]. \square

Proposition 4.2. *Let X be a smooth proper rigid space over K .*

1. *Let Y be any smooth affinoid rigid space. Then*

$$H_{\text{ét}}^n(X \times Y, \mathcal{O}) = H_{\text{ét}}^n(X, \mathcal{O}) \otimes_K \mathcal{O}(Y).$$

2. *Let Y be an affinoid perfectoid space over K . Then there are natural isomorphisms:*

$$(i) \quad H_{\text{ét}}^n(X \times Y, \mathcal{O}) = H_{\text{ét}}^n(X, \mathcal{O}) \otimes_K \mathcal{O}(Y),$$

$$(ii) \quad H_v^n(X \times Y, \mathcal{O}) = H_v^n(X, \mathcal{O}) \otimes_K \mathcal{O}(Y),$$

$$(iii) \quad H_v^n(X \times Y, \mathcal{O}^+/p^k) \stackrel{a}{=} H_v^n(X, \mathcal{O}^+/p^k) \otimes_{K^+} \mathcal{O}^+/p^k(Y).$$

In particular, $R^n \pi_{\text{ét}}^{\diamond} \mathcal{O} = H_{\text{ét}}^n(X, \mathcal{O}) \otimes_K \mathcal{O}$ and $R^n \pi_{v*}^{\diamond} \mathcal{O} = H_v^n(X, \mathcal{O}) \otimes_K \mathcal{O}$.*

3. *Let Y be an affinoid perfectoid space over K and assume that K is algebraically closed. Then the following natural map is an almost isomorphism:*

$$H_{\text{ét}}^n(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}^+(Y) \rightarrow H_v^n(X \times Y, \mathcal{O}^+).$$

We remark that these statements are all easier special cases of a much more general adic version of Grothendieck’s “cohomology and base-change” which will be proved in the sequel [17, Theorem 3.18]. For example, smoothness is not necessary for Proposition 4.2.1.

Proof. We start with part 2.(ii): Since X is quasi-compact separated, we can choose a finite cover \mathcal{U} of X by affinoids U_i with affinoid intersections that are étale over a torus, and thus admit toric pro-finite-étale covers $\tilde{U}_i \rightarrow U_i$. Then any fibre product of the \tilde{U}_i over X is affinoid perfectoid. Consequently, the cohomology $H_v^n(X, \mathcal{O}^+)$ is almost computed by the Čech complex $\check{C}^\bullet(\tilde{\mathcal{U}}, \mathcal{O}^+)$ where $\tilde{\mathcal{U}}$ is the pro-étale cover of X by the \tilde{U}_i .

Each $H_v^n(X, \mathcal{O}^+)$ has bounded p -torsion: This follows from the fact that the sheaves in Proposition 4.3 are coherent, so $H_v^n(X, \mathcal{O})$ is finite dimensional. By an application of Lemma A.3.1 in the appendix, this implies that $\check{C}_v^\bullet(\tilde{\mathcal{U}}, \mathcal{O}^+)$ is a complex of p -torsionfree p -complete K^+ -modules whose cohomology has bounded p -torsion.

We now add the factor Y : Clearly the $\tilde{U}_i \times Y$ form a cover $\tilde{\mathcal{U}} \times Y$ of $X \times Y$ such that all spaces appearing in the Čech nerve are still affinoid perfectoid. Let $\mathcal{O}^+(Y) = S^+$, then the fact that $\check{C}_v^\bullet(\tilde{\mathcal{U}}, \mathcal{O}^+)$ has cohomology of bounded torsion implies by Lemma A.3.2 that

$$\begin{aligned} H_v^n(X \times Y, \mathcal{O}^+) &\stackrel{a}{=} H^n(\check{C}^\bullet(\tilde{\mathcal{U}} \times Y, \mathcal{O}^+)) \stackrel{a}{=} H^n(\check{C}^\bullet(\tilde{\mathcal{U}}, \mathcal{O}^+) \hat{\otimes}_{K^+} S^+) \\ &\stackrel{a}{=} H^n(\check{C}^\bullet(\tilde{\mathcal{U}}, \mathcal{O}^+)) \hat{\otimes}_{K^+} S^+ \\ &\stackrel{a}{=} H_v^n(X, \mathcal{O}^+) \otimes_{K^+} S^+. \end{aligned}$$

After inverting p , this gives the desired equality for 2.(ii).

Part 2.(iii) also follows from the displayed equation by comparing the long exact sequences of $0 \rightarrow \mathcal{O}^+ \rightarrow \mathcal{O}^+ \rightarrow \mathcal{O}^+/p^k \rightarrow 0$ for $X \times Y$ and X , using the 5-Lemma.

If K is algebraically closed, we also deduce part 3 using the Primitive Comparison Theorem, [30, Theorem 5.1].

Part 2.(i) follows by a similar argument using instead the cover \mathcal{U} : By Lemma 4.1, the group $\check{H}^n(\mathcal{U}, \mathcal{O} \hat{\otimes} S)$ computes $H_{\text{ét}}^n(X \times Y, \mathcal{O})$. Since each $\check{H}^n(\mathcal{U}, \mathcal{O})$ is finite, we can now again apply Lemma A.3.2 to the complex of K^+ -modules $\check{C}^\bullet(\mathcal{U}, \mathcal{O}^+)$ to see that:

$$H_{\text{ét}}^n(X \times Y, \mathcal{O}) = H^n(\check{C}^\bullet(\mathcal{U}, \mathcal{O}^+) \hat{\otimes}_{K^+} S^+)[\frac{1}{p}] = \check{H}^n(\mathcal{U}, \mathcal{O}^+) \hat{\otimes}_{K^+} S^+[\frac{1}{p}] = H_{\text{ét}}^n(X, \mathcal{O}) \otimes S.$$

Part 1 can be seen similarly: By Tate acyclicity, $\check{H}^n(\mathcal{U} \times Y, \mathcal{O})$ computes $H^n(X \times Y, \mathcal{O})$. For any affinoid $U \subseteq X$, the map $\mathcal{O}^+(U) \hat{\otimes}_{K^+} \mathcal{O}^+(Y) \rightarrow \mathcal{O}^+(U \times Y)$ has bounded p -torsion cokernel since $U \times Y$ is uniform. Hence in the composition

$$H^n(\check{C}^\bullet(\mathcal{U}, \mathcal{O}^+) \hat{\otimes}_{K^+} \mathcal{O}^+(Y)) \rightarrow H^n(\check{C}^\bullet(\mathcal{U}, \mathcal{O}^+) \hat{\otimes}_{K^+} \mathcal{O}^+(Y)) \rightarrow H^n(\check{C}^\bullet(\mathcal{U} \times Y, \mathcal{O}^+)),$$

the second map becomes an isomorphism after inverting p , while the first map is an isomorphism by Lemma A.3. After inverting p , this gives the desired statement. \square

Second, we need the following result of Scholze:

Proposition 4.3 ([31, Proposition 3.23], [14, 2.24–2.25]). *Let X be any smooth rigid space and let $\nu : X_v \rightarrow X_{\text{ét}}$ be the natural morphism of sites. Then $R^n \nu_* \mathcal{O} = \wedge^n \tilde{\Omega}_X^1$.*

Proof of Proposition 2.4. Using Proposition 4.3, we see that the 5-term exact sequence of the Leray sequence for the morphism ν is of the form

$$(6) \quad 0 \rightarrow H_{\text{ét}}^1(X, \mathcal{O}) \rightarrow H_v^1(X, \mathcal{O}) \rightarrow H^0(X, \tilde{\Omega}_X^1) \xrightarrow{\partial_X} H_{\text{ét}}^2(X, \mathcal{O}) \xrightarrow{j_X} H_v^2(X, \mathcal{O}).$$

If K is algebraically closed, it follows from the degeneration of the Hodge–Tate spectral sequence [1, Theorem 1.7.(ii)] that $\partial_X = 0$. This implies that the fourth map j_X is injective.

The general case follows from this: It suffices to prove that $\partial_X = 0$, or equivalently that j_X is injective. Let C be the completion of an algebraic closure of K . By Proposition 4.2 for $Y = \text{Spa}(C)$, the base-change of j_X along $K \rightarrow C$ admits an identification $j_X \otimes_K C = j_{X_C}$. This is injective by the algebraically closed case, hence j_X is injective. \square

We now move on to the relative Hodge–Tate sequence:

Proof of Proposition 2.6. The first part follows from comparing Proposition 4.2.1 and 2.

To see the second part, we tensor the Hodge–Tate sequence for X from Proposition 2.4 with \mathcal{O} and see from Proposition 4.2.2.(i) and (ii) for $i = 1$ that we obtain identifications

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^1 \pi_{\text{ét}*}^\diamond \mathcal{O} & \longrightarrow & R^1 \pi_{v*}^\diamond \mathcal{O} & & \\ & & \uparrow \wr & & \uparrow \wr & & \\ 0 & \longrightarrow & H_{\text{ét}}^1(X, \mathcal{O}) \otimes \mathcal{O} & \longrightarrow & H_v^1(X, \mathcal{O}) \otimes \mathcal{O} & \longrightarrow & H^0(X, \tilde{\Omega}^1(X)) \otimes \mathcal{O} \longrightarrow 0. \quad \square \end{array}$$

We can now also address Lemma 2.16.(F):

Proposition 4.4. *The map $(R^2 \pi_{\text{ét}*} \mathcal{O})^\diamond \rightarrow R^2 \pi_{v*}^\diamond \mathcal{O}$ is injective.*

Proof. In the notation of the sequence (6) above, Proposition 4.2.2 for $i = 2$ identifies this map with $j_X \otimes \mathcal{O} : H_{\text{ét}}^2(X, \mathcal{O}) \otimes \mathcal{O} \hookrightarrow H_v^2(X, \mathcal{O}) \otimes \mathcal{O}$. This is injective by Proposition 2.6. \square

From the case of $i = 0$ of Proposition 4.2, we will moreover deduce part (D) of Lemma 2.16 (see the end of this subsection). For this we will use the following consequence:

Corollary 4.5. *Suppose that X is geometrically connected. Then*

$$(\pi_{\text{ét}*} \mathcal{O})^\diamond = \mathcal{O} = \pi_{\text{ét}*}^\diamond \mathcal{O} = \pi_{v*}^\diamond \mathcal{O}.$$

The analogous statements hold for \mathcal{O}_1^\times , \mathcal{O}^\times and \mathcal{O}^{+a}/p^k .

Proof. The first part follows from Proposition 4.2.1-2.(i): Here we use that $H^0(X, \mathcal{O}) = K$ since X is geometrically connected. The cases of \mathcal{O}^\times and \mathcal{O}_1^\times follow as these are subsheaves of \mathcal{O} .

For \mathcal{O}^{+a}/p^k , we first recall that we have $(\pi_{\text{ét}*}(\mathcal{O}^+/p^k))^\diamond = \pi_{\text{ét}*}^\diamond(\mathcal{O}^+/p^k)$ by Corollary 3.16. Second, we have $\pi_{\text{ét}*}^\diamond(\mathcal{O}^+/p^k) = \pi_{v*}^\diamond(\mathcal{O}^+/p^k)$ by Lemma 3.14. Finally, it follows from Proposition 4.2.2.(iii) that $\pi_{v*}^\diamond(\mathcal{O}^+/p^k) = H_v^0(X, \mathcal{O}^+/p^k) \otimes \mathcal{O}^+/p^k$. It thus remains to see that $H_v^0(X, \mathcal{O}^+/p^k) \stackrel{a}{=} K^+/p^k$. For this we can use Proposition 3.2 to reduce to the case that K is algebraically closed, where the statement follows from Proposition 4.2.3. \square

We can also deduce a version of the Primitive Comparison Theorem relatively over Y :

Corollary 4.6. *Assume that K is algebraically closed. Let X be a smooth proper rigid space over K and let Y be affinoid perfectoid over K . Then the natural map*

$$H_v^n(X, \mathbb{Z}/p^k) \otimes \mathcal{O}^+(Y)/p^k \rightarrow H_v^n(X \times Y, \mathcal{O}^+/p^k)$$

is an almost isomorphism for all $n \geq 0$. In particular, the natural map

$$H_v^n(X, \mathbb{F}_p) \otimes \mathcal{O}^{b+}(Y) \rightarrow H_v^n(X \times Y, \mathcal{O}^{b+})$$

is an almost isomorphism for all $n \geq 0$, compatible with Frobenius actions on both sides.

Proof. The first part follows from Proposition 4.2.3, using the sequence $\mathcal{O}^+ \rightarrow \mathcal{O}^+ \rightarrow \mathcal{O}^+/p^k$ and the fact that $H_{\text{ét}}^n(X, \mathbb{Z}/p^k) = H_v^n(X, \mathbb{Z}/p^k)$ by [32, Propositions 14.7 and 14.8]. The second part follows from the case of $k = 1$ in the inverse limit over Frobenius. \square

Proposition 4.7 (Künneth formula). *Let X be a smooth proper rigid space and let Y be affinoid perfectoid. Then there is a natural isomorphism for all $n \geq 0$*

$$H_v^n(X \times Y, \mathbb{F}_p) = \left(H_v^{n-1}(X, \mathbb{F}_p) \otimes H_v^1(Y, \mathbb{F}_p) \right) \oplus \left(H_v^n(X, \mathbb{F}_p) \otimes H_v^0(Y, \mathbb{F}_p) \right).$$

Proof. We consider the v -cohomological long exact sequence for the Artin–Schreier sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}^b \xrightarrow{\text{AS}} \mathcal{O}^b \rightarrow 0$$

on $X \times Y$. By Corollary 4.6.2, this yields a long exact sequence

$$\dots \xrightarrow{\text{AS}} H^{n-1}(X, \mathbb{F}_p) \otimes \mathcal{O}^b(Y) \rightarrow H_v^n(X \times Y, \mathbb{F}_p) \rightarrow H^n(X, \mathbb{F}_p) \otimes \mathcal{O}^b(Y) \xrightarrow{\text{AS}} \dots$$

Since $H_v^n(Y, \mathcal{O}) = 0$ for $n \geq 1$, we have $H_v^1(Y, \mathbb{F}_p) = \text{coker}(\text{AS} : \mathcal{O}^b(Y) \rightarrow \mathcal{O}^b(Y))$ and $H_v^n(Y, \mathbb{F}_p) = 0$ for $n \geq 2$. It follows that we can rewrite the above as a natural extension

$$0 \rightarrow H_v^{n-1}(X, \mathbb{F}_p) \otimes H_v^1(Y, \mathbb{F}_p) \rightarrow H_v^n(X \times Y, \mathbb{F}_p) \rightarrow H_v^n(X, \mathbb{F}_p) \otimes H_v^0(Y, \mathbb{F}_p) \rightarrow 0.$$

Recall that $\pi_0(Y)$ is always a profinite space [5, Tag 0906]. By comparing to the case that $Y = \pi_0(Y)$ is strictly totally disconnected, in which case $H_v^1(Y, \mathbb{F}_p) = 0$, we see that pullback along $\overline{X \times Y} \rightarrow X \times \pi_0(Y)$ defines a natural splitting of the last map. \square

We use this to complete the second part of Lemma 2.14.(E):

Corollary 4.8. *For any $N \in \mathbb{N}$ and $n \geq 0$, we have a natural isomorphism*

$$R^n \pi_{\text{ét}*}^\diamond \mathbb{Z}/N\mathbb{Z} = R^n \pi_{v*}^\diamond \mathbb{Z}/N\mathbb{Z}.$$

If K is algebraically closed, this is isomorphic to $H_{\text{ét}}^n(X, \mathbb{Z}/N\mathbb{Z})$, the locally constant sheaf on $\text{Perf}_{K,v}$ associated to the group $H_{\text{ét}}^n(X, \mathbb{Z}/N\mathbb{Z})$.

Proof. For N coprime to p , this follows from general base-change results for the diagram

$$\begin{array}{ccc} X_v^\diamond & \xrightarrow{\pi_v^\diamond} & \text{Perf}_{K,v} \\ \downarrow & & \downarrow \nu \\ X_{\text{ét}}^\diamond & \xrightarrow{\pi_{\text{ét}}^\diamond} & \text{Perf}_{K,\text{ét}}, \end{array}$$

namely by [32, Theorem 16.1.(iii) and Proposition 16.6], the base-change morphism

$$\nu^* R^n \pi_{\text{ét}*}^\diamond \mathbb{Z}/N\mathbb{Z} \rightarrow R^n \pi_{v*}^\diamond \mathbb{Z}/N\mathbb{Z}$$

is an isomorphism, and $\nu_* \nu^* F = F$ for any sheaf on $\text{Perf}_{K,\text{ét}}$ by [32, Proposition 14.7].

The last sentence of the Corollary is clear when $K^+ = \mathcal{O}_K$ since any sheaf on $\text{Spa}(K, \mathcal{O}_K)_{\text{ét}}$ is constant. The general case follows from this: Let $j : \text{Spa}(K, \mathcal{O}_K) \hookrightarrow \text{Spa}(K, K^+)$ be the natural open immersion, then it follows from [20, Proposition 8.1.2.(ii)] that

$$R^n \pi_{\text{ét}*} \mathbb{Z}/N\mathbb{Z} = j_* j^* R^n \pi_{\text{ét}*} \mathbb{Z}/N\mathbb{Z} = j_* \underline{H_{\text{ét}}^n(X, \mathbb{Z}/N\mathbb{Z})} = \underline{H_{\text{ét}}^n(X, \mathbb{Z}/N\mathbb{Z})}.$$

For N a power of p , we can reduce by induction to the case of $N = p$. Then $R^n \pi_{v*}^\diamond \mathbb{F}_p$ is the v -sheafification of

$$Y \mapsto H_v^n(X \times Y, \mathbb{F}_p).$$

By Proposition 4.7, this is the locally constant sheaf of $H_{\text{ét}}^n(X, \mathbb{F}_p) = H_v^n(X, \mathbb{F}_p)$. \square

At this point, we can complete the proof of Lemma 2.16:

Corollary 4.9. *For any $n \geq 0$, the following morphisms are isomorphisms:*

$$(R^n \pi_{\text{ét}*} \mu_{p^\infty})^\diamond \rightarrow R^n \pi_{\text{ét}*}^\diamond \mu_{p^\infty} \rightarrow R^n \pi_{v*}^\diamond \mu_{p^\infty}$$

Proof. By quasi-compactness, it suffices to prove this for μ_{p^∞} replaced by μ_{p^m} . We can check the statement locally on $\text{Perf}_{K,\text{ét}}$, and may therefore assume that K contains $\mu_{p^m}(\overline{K})$. Then $\mu_{p^m} \cong \underline{\mathbb{Z}/p^m\mathbb{Z}}$ and the statement follows from Corollary 3.19 and Corollary 4.8. \square

Proof of Lemma 2.16. For Part (D), we use that by Corollary 4.5, the left-exact sequence

$$0 \rightarrow \pi_{\text{ét}*}^\diamond \mu_{p^\infty} \rightarrow \pi_{\text{ét}*}^\diamond \mathcal{O}_1^\times \rightarrow \pi_{\text{ét}*}^\diamond \mathcal{O} \rightarrow 0$$

is also right-exact. The analogous statements hold for $\pi_{v*}^\diamond \mathcal{O}$ and $(\pi_{\text{ét}*} \mathcal{O})^\diamond$. This implies Lemma 2.16.(D). Part (E) is Corollary 4.9, (F) is Proposition 4.4. \square

4.2. Cohomology of $\overline{\mathcal{O}}^\times$. We now move on to proving the remaining parts of Lemma 2.14 concerning the sheaf $\overline{\mathcal{O}}^\times$. For some relevant background information on this sheaf, the interested reader might find it helpful to look at [14, §2.3-§2.4] on which some arguments in the following are based. We begin with some preparations.

Lemma 4.10. *Let X be a rigid space over an algebraically closed field K . Then evaluation at points in $X(K)$ induces a unique injective map fitting into the commutative diagram:*

$$\begin{array}{ccc} \mathcal{O}^\times(X) & \longrightarrow & \text{Map}_{\text{cts}}(X(K), K^\times) \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}^\times(X) & \hookrightarrow & \text{Map}_{\text{lc}}(X(K), K^\times/(1 + \mathfrak{m})). \end{array}$$

Proof. The first arrow is given by interpreting $f \in \mathcal{O}^\times(X)$ as a morphism $X \rightarrow \mathbb{G}_m$ and evaluating on K -points. By the Maximum Modulus Principle, this sends $f \in \mathcal{O}^\times(X)$ into $\text{Map}_{\text{cts}}(X(K), 1 + \mathfrak{m})$ if and only if $f \in \mathcal{O}_1^\times(X)$. It now suffices to construct the bottom map for affinoid X , where $\overline{\mathcal{O}}^\times(X) = \mathcal{O}^\times[\frac{1}{p}](X)/\mathcal{O}_1^\times(X)$ by [14, Lemma 2.19] \square

Lemma 4.11. *Let X be a smooth proper rigid space over K that is geometrically connected.*

1. *For Y any reduced rigid space over K , we have $\overline{\mathcal{O}}^\times(X \times Y) = \overline{\mathcal{O}}^\times(Y)$.*
2. *For Y any perfectoid space over K , we have $\overline{\mathcal{O}}^\times(X \times Y) = \overline{\mathcal{O}}^\times(Y)$.*

In particular, we have $(\pi_{\acute{e}t}\overline{\mathcal{O}}^\times)^\diamond = \pi_{\acute{e}t*}^\diamond\overline{\mathcal{O}}^\times = \pi_{v*}^\diamond\overline{\mathcal{O}}^\times = \overline{\mathcal{O}}^\times$.*

Proof. We first explain how to deduce part 2 from part 1: The statement is local on Y , so we can assume that Y is affinoid perfectoid. By Proposition 3.17, we can then find an inverse system of affinoid smooth rigid spaces $(Y_i)_{i \in I}$ over K such that $Y \approx \varprojlim Y_i$. Assuming part 1, we then have by Proposition 3.2:

$$\overline{\mathcal{O}}^\times(X \times Y) = \varinjlim_{i \in I} \overline{\mathcal{O}}^\times(X \times Y_i) = \varinjlim_{i \in I} \overline{\mathcal{O}}^\times(Y_i) = \overline{\mathcal{O}}^\times(Y).$$

Next, let us explain the last sentence of the Lemma: Recall that we had already seen in Corollary 3.16 that $(\pi_{\acute{e}t*}\overline{\mathcal{O}}^\times)^\diamond = \pi_{\acute{e}t*}^\diamond\overline{\mathcal{O}}^\times$. That $\pi_{\acute{e}t*}^\diamond\overline{\mathcal{O}}^\times = \pi_{v*}^\diamond\overline{\mathcal{O}}^\times$ follows from Lemma 3.14. Part 2 implies $\pi_{v*}^\diamond\overline{\mathcal{O}}^\times = \overline{\mathcal{O}}^\times$ immediately from the definition.

It thus remains to prove part 1. For this, we can assume that Y is affinoid and connected. Second, we can without loss of generality assume that K is algebraically closed: Let \overline{K} be an algebraic closure of K and let C be its completion. Then to deduce the general case from that over C , let $G := \text{Gal}(\overline{K}|K)$ and consider the G -torsor $X \times Y_C \rightarrow X \times Y$. Assuming part 1 for C , and using that $\overline{\mathcal{O}}^\times$ is a v -sheaf by Lemma 3.14, we then have

$$\overline{\mathcal{O}}^\times(X \times Y) = \overline{\mathcal{O}}^\times(X \times Y_C)^G = \overline{\mathcal{O}}^\times(Y_C)^G = \overline{\mathcal{O}}^\times(Y).$$

Now for algebraically closed K , we start with $Y = \text{Spa}(K)$. In this case, using that X is connected, we compare to the universal pro-étale cover $\tilde{X} \rightarrow X$ of [14, §4]: Using the exponential sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}^\times[\frac{1}{p}] \rightarrow \overline{\mathcal{O}}^\times \rightarrow 1$ from [14, Lemma 2.18], we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^\times[\frac{1}{p}](\tilde{X}) & \longrightarrow & \overline{\mathcal{O}}^\times(\tilde{X}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}) = 0 \\ \uparrow & & \updownarrow & & \uparrow \\ \mathcal{O}^\times[\frac{1}{p}](X) & \longrightarrow & \overline{\mathcal{O}}^\times(X) & \longrightarrow & H^1(X, \mathcal{O}) \end{array}$$

in which by [14, Proposition 3.10], the top row can be identified with the sequence

$$K^\times[\frac{1}{p}] \rightarrow K^\times/(1 + \mathfrak{m}) \rightarrow 1.$$

In particular, the first vertical arrow is surjective. The second vertical arrow is injective since $\tilde{X} \rightarrow X$ is a Galois cover. This shows that $\overline{\mathcal{O}}^\times(X) = \overline{\mathcal{O}}^\times(\tilde{X}) = K^\times/(1 + \mathfrak{m})$. In particular, the boundary map $\overline{\mathcal{O}}^\times(X) \rightarrow H^1(X, \mathcal{O})$ vanishes.

We now move on to the case of general Y : We will show that the boundary map ∂ of the exponential sequence

$$0 \rightarrow H^0(X \times Y, \mathcal{O}) \rightarrow H^0(X \times Y, \mathcal{O}^\times[\frac{1}{p}]) \rightarrow H^0(X \times Y, \overline{\mathcal{O}}^\times) \xrightarrow{\partial} H_v^1(X \times Y, \mathcal{O})$$

vanishes as well. This implies the desired result: We already know from Corollary 4.5 that the first two terms identify with $\mathcal{O}(Y)$ and $\mathcal{O}^\times(Y)[\frac{1}{p}]$. Using Lemma 3.14 again, we can then compute in the étale topology that their quotient is $\overline{\mathcal{O}}^\times(Y)$ because $H_{\acute{e}t}^1(Y, \mathcal{O}) = 0$.

To see that $\partial = 0$, we can without loss of generality assume that $(K, K^+) = (K, \mathcal{O}_K)$: Indeed, pullback along $\text{Spa}(K, \mathcal{O}_K) \rightarrow \text{Spa}(K, K^+)$ only changes the integral subrings, so the comparison map $H_v^1(X \times Y, \mathcal{O}) \rightarrow H_v^1(X \times Y \times_{\text{Spa}(K, K^+)} \text{Spa}(K, \mathcal{O}_K), \mathcal{O})$ is an isomorphism.

The idea is now to compare the boundary map of the exponential sequence for X and $X \times Y$ via the pullback along $X \rightarrow X \times Y$ for points in $Y(K)$. This results in a commutative diagram:

$$\begin{array}{ccc} H^0(X \times Y, \overline{\mathcal{O}}^\times) & \longrightarrow & H_v^1(X \times Y, \mathcal{O}) \\ \downarrow & & \downarrow \\ \text{Map}(Y(K), \overline{\mathcal{O}}^\times(X)) & \longrightarrow & \text{Map}(Y(K), H_v^1(X, \mathcal{O})) \end{array}$$

By Proposition 4.2, we know that

$$H_v^1(X \times Y, \mathcal{O}) = H_{\text{ét}}^1(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}(Y).$$

This shows that the right vertical map can be identified with $H^1(X, \mathbb{Q}_p)$ tensored with the evaluation map $s : \mathcal{O}(Y) \rightarrow \text{Map}_{\text{cts}}(Y(K), K)$. This is injective since Y is a reduced classical rigid space, thus the right vertical map is injective. But the bottom map is $= 0$ by the case of $Y = \text{Spa}(K)$. Hence the top map vanishes, as we wanted to see. \square

Finally, we turn to the remaining part of Lemma 2.14 (B), about the top morphism. We need to compare étale and v -cohomology of $\overline{\mathcal{O}}^\times$ on products of X with perfectoid spaces:

Proposition 4.12. *Let $\mathcal{F} = \mathcal{O}^{+a}/p$ and $n \in \mathbb{N}$; or $\mathcal{F} = \overline{\mathcal{O}}^\times$ and $n \in \{0, 1\}$.*

1. *Let X be a smooth qcqs rigid space and let Y be an affinoid perfectoid space over K . Then*

$$H_{\text{ét}}^n(X \times Y, \mathcal{F}) = H_v^n(X \times Y, \mathcal{F}).$$

2. *Let Y be any spatial diamond over K . Let $\tilde{X} = \varprojlim_{i \in I} X_i$ be a diamond which is a limit of smooth qcqs rigid spaces over K with finite étale transition maps. Then*

$$\varinjlim_i H_v^n(X_i \times Y, \mathcal{F}) = H_v^n(\tilde{X} \times Y, \mathcal{F}).$$

Part 1 is proved in much greater generality in [12, Proposition 2.14], but we will in the next section also need part 2, from which it is easy to deduce part 1 independently.

Proof. The proof will be completed in several steps:

Step 1. We first observe that if Y is any qcqs perfectoid space and $S = \varprojlim S_i$ is a profinite space, then by Proposition 3.2, we can apply Lemma 3.8 to $S \times Y \sim \varprojlim S_i \times Y$ to see that

$$H_{\text{ét}}^n(S \times Y, \mathcal{F}) = \varinjlim H_{\text{ét}}^n(S_i \times Y, \mathcal{F}) = \varinjlim \text{Map}(S_i, H_{\text{ét}}^n(Y, \mathcal{F})).$$

Step 2. When Y is affinoid perfectoid, then we have

$$\varinjlim H_{\text{ét}}^n(X_i \times Y, \mathcal{F}) = H_v^n(\tilde{X} \times Y, \mathcal{F}).$$

Proof. Choose any element $0 \in I$. By a Čech-argument, it suffices to prove the statement after replacing X_0 by a qcqs open U that admits a perfectoid (say, toric) cover $X_{\infty,0} \sim \varprojlim X_{j,0} \rightarrow X_0$ with pro-finite-étale Galois group $G = \varprojlim G_j$.

Let $X_{j,i} := X_{j,0} \times_{X_0} X_i$ and to simplify notation let $Z_{j,i} := X_{j,i} \times Y$. Furthermore, let

$$\tilde{Z}_j := \tilde{X} \times_{X_0} Z_{j,0} = \varprojlim_i Z_{j,i}.$$

In this notation, our space $\tilde{X} \times Y$ is \tilde{Z}_0 . In summary, we have a commutative diagram

$$\begin{array}{ccc} \tilde{Z}_\infty & \longrightarrow & Z_{\infty,0} \\ \downarrow & & \downarrow \\ \tilde{Z}_0 & \longrightarrow & Z_{0,0} \end{array}$$

in which the left map is a pro-finite-étale G -torsor under a perfectoid space, and the top morphism is a pro-finite-étale morphism of perfectoid spaces.

Since \tilde{Z}_∞ is perfectoid we have $H_v^n(\tilde{Z}_\infty, \mathcal{O}^+/p) \stackrel{a}{=} H_{\text{ét}}^n(\tilde{Z}_\infty, \mathcal{O}^+/p)$, and similarly for $\overline{\mathcal{O}}^\times$ by the exponential sequence of [14, Lemma 2.18]. We endow this with the discrete topology. By Step 1, we then have

$$H_{\text{ét}}^n(\tilde{Z}_\infty \times G^k, \mathcal{O}^+/p) = \text{Map}_{\text{cts}}(G^k, H_{\text{ét}}^n(\tilde{Z}_\infty, \mathcal{O}^+/p)).$$

It follows that the Čech-to-sheaf spectral sequence of $\tilde{Z}_\infty \rightarrow \tilde{Z}_0$ is a Cartan–Leray spectral sequence in the almost category

$$H_{\text{cts}}^n(G, H_{\text{ét}}^m(\tilde{Z}_\infty, \mathcal{O}^+/p)) \Rightarrow H_v^{n+m}(\tilde{Z}_0, \mathcal{O}^+/p).$$

Since $\tilde{Z}_\infty = \varprojlim Z_{\infty,i} \rightarrow Z_{\infty,0}$ is a pro-finite-étale morphism of perfectoid spaces, we have

$$H_{\text{ét}}^m(\tilde{Z}_\infty, \mathcal{O}^+/p) = \varinjlim_i H_{\text{ét}}^m(Z_{\infty,i}, \mathcal{O}^+/p) = \varinjlim_i \varinjlim_j H_{\text{ét}}^m(Z_{j,i}, \mathcal{O}^+/p)$$

by Proposition 3.2. We deduce by [28, Proposition 1.2.5] that we now have for any $n, m \geq 0$:

$$H_{\text{cts}}^n(G, H_{\text{ét}}^m(\tilde{Z}_\infty, \mathcal{O}^+/p)) = \varinjlim_i \varinjlim_j H^n(G_j, H_{\text{ét}}^m(Z_{j,i}, \mathcal{O}^+/p)).$$

But these are the terms appearing in the usual étale Cartan–Leray sequence for $Z_{j,i} \rightarrow Z_{0,i}$:

$$H^n(G_j, H_{\text{ét}}^m(Z_{j,i}, \mathcal{O}^+/p)) \Rightarrow H_{\text{ét}}^{n+m}(Z_{0,i}, \mathcal{O}^+/p).$$

Thus the abutment of the first sequence is $= \varinjlim_i H_{\text{ét}}^{n+m}(Z_{0,i}, \mathcal{O}^+/p)$, as we wanted to see.

The case of $\overline{\mathcal{O}}^\times$ is similar, but instead using only the 5-term exact sequence

$$0 \rightarrow H_{\text{cts}}^1(G, \overline{\mathcal{O}}^\times(\tilde{Z}_\infty)) \rightarrow H_v^1(\tilde{Z}_0, \overline{\mathcal{O}}^\times) \rightarrow H_{\text{ét}}^1(\tilde{Z}_\infty, \overline{\mathcal{O}}^\times)^G \rightarrow H_{\text{cts}}^2(G, \overline{\mathcal{O}}^\times(\tilde{Z}_\infty))$$

which by the above arguments is the colimit over i and j of

$$0 \rightarrow H_{\text{cts}}^1(G_j, \overline{\mathcal{O}}^\times(Z_{j,i})) \rightarrow H_{\text{ét}}^1(Z_{0,i}, \overline{\mathcal{O}}^\times) \rightarrow H_{\text{ét}}^1(Z_{j,i}, \overline{\mathcal{O}}^\times)^{G_j} \rightarrow H_{\text{cts}}^2(G_j, \overline{\mathcal{O}}^\times(Z_{j,i})).$$

This finishes the proof of Step 2. \square

Step 3. Part 1 holds.

Proof. This follows from Step 2 as the special case of $\tilde{X} = X$. \square

Step 4. Part 2 holds for affinoid perfectoid Y .

Proof. It follows from part 1 that the left hand side of the statement of part 2 is equal to $\varinjlim H_{\text{ét}}^n(X_i \times Y, \mathcal{F})$. Thus this follows from Step 2. \square

Step 5. Part 2 holds for general spatial diamonds.

Proof. Any spatial diamond admits a cover $\tilde{Y} \rightarrow Y$ by an affinoid perfectoid space \tilde{Y} such that all finite products $\tilde{Y} \times_Y \cdots \times_Y \tilde{Y}$ are affinoid perfectoid (e.g use [32, Propositions 11.5, 11.14 and Lemma 7.19]). Comparing the Čech-to-sheaf spectral sequence

$$\check{H}^n(\tilde{Y} \rightarrow Y, H_v^m(X_i \times -, \mathcal{F})) \Rightarrow H_v^{n+m}(X_i \times Y, \mathcal{F})$$

to the sequence for X_i replaced by \tilde{X} , we deduce the result from Step 4. \square

This finishes the proof of Proposition 4.12. \square

Towards Lemma 2.14, we can use this to describe the cohomology sheaves:

Lemma 4.13. *Let X, Y be locally spatial diamonds over K with $X(K) \neq \emptyset$. Let τ be either the étale or the v -topology and let F be a τ -sheaf on LSD_K such that the pullback $F \rightarrow \pi_* F$ of sheaves on Y_τ along $\pi : X \times Y \rightarrow Y$ is an isomorphism. Then the Leray sequence*

$$0 \rightarrow H_\tau^1(Y, \pi_* F) \rightarrow H_\tau^1(X \times Y, F) \rightarrow R^1 \pi_{\tau*} F(Y) \rightarrow 1$$

is a short exact sequence.

Proof. This is a standard argument that we learned from Gabber's simplification of [8, Lemma 5]: The full Leray 5-term exact sequence is of the form

$$0 \rightarrow H_\tau^1(Y, \pi_* F) \rightarrow H_\tau^1(X \times Y, F) \rightarrow R^1 \pi_* F(Y) \rightarrow H_\tau^2(Y, \pi_* F) \rightarrow H_\tau^2(X \times Y, F).$$

By assumption, the last map agrees with the pullback map

$$\pi^* : H_\tau^2(Y, F) \rightarrow H_\tau^2(X \times Y, F).$$

Any point $x \in X(K)$ now defines a splitting of π^* , showing that this map is injective. \square

Proof of Lemma 2.14.(A)-(B). For part (A), we consider the a priori left exact sequence

$$1 \rightarrow \pi_{\text{ét}*}^\diamond \mathcal{O}_1^\times \rightarrow \pi_{\text{ét}*}^\diamond \mathcal{O}^\times \rightarrow \pi_{\text{ét}*}^\diamond \overline{\mathcal{O}}^\times \rightarrow 1.$$

By Corollary 4.5 and Lemma 4.11, this gets identified with

$$1 \rightarrow \mathcal{O}_1^\times \rightarrow \mathcal{O}^\times \rightarrow \overline{\mathcal{O}}^\times \rightarrow 1,$$

which is short exact. Hence the boundary map $\pi_{\text{ét}*}^\diamond \overline{\mathcal{O}}^\times \rightarrow R^1 \pi_{\text{ét}*}^\diamond \mathcal{O}_1^\times$ vanishes. This shows the statement for the middle row. The other rows are completely analogous.

The first part of (B) was Corollary 3.16. To finish the proof of (B) it remains to prove that the map

$$R^1 \pi_{\text{ét}*}^\diamond \overline{\mathcal{O}}^\times \rightarrow R^1 \pi_{v*}^\diamond \overline{\mathcal{O}}^\times$$

is an isomorphism. We may prove this locally on $\text{Perf}_{K, \text{ét}}$, and may therefore assume that $X(K) \neq \emptyset$. By Lemma 4.11, we can then apply Lemma 4.13 to get an exact sequence

$$1 \rightarrow H_v^1(Y, \overline{\mathcal{O}}^\times) \rightarrow H_v^1(X \times Y, \overline{\mathcal{O}}^\times) \rightarrow R^1 \pi_{v*}^\diamond \overline{\mathcal{O}}^\times(Y) \rightarrow 1.$$

It also applies for the étale topology, so we also get a short exact sequence

$$1 \rightarrow H_{\text{ét}}^1(Y, \overline{\mathcal{O}}^\times) \rightarrow H_{\text{ét}}^1(X \times Y, \overline{\mathcal{O}}^\times) \rightarrow R^1 \pi_{\text{ét}*}^\diamond \overline{\mathcal{O}}^\times(Y) \rightarrow 1.$$

The first two terms of these sequences are isomorphic via the natural maps by Proposition 4.12.1. Thus the third terms are isomorphic. \square

5. PROOF OF MAIN THEOREM

At this point we have completed the proof of Lemma 2.16 and of (A)–(B) of Lemma 2.14.

We are left to prove Lemma 2.14.(C) and to explain how to deduce Proposition 2.15, which is not completely formal from the diagram. Finally, we need to prove Corollary 2.9.

We can assume that X is connected. Fix a base point $x \in X(\overline{K})$. We can then define the universal pro-finite-étale cover from [14, §3.4]: This is the diamond \tilde{X} over K defined as

$$\tilde{\pi} : \tilde{X} := \varprojlim_{X' \rightarrow X} X' \rightarrow \text{Spd}(K)$$

where the limit ranges over connected finite étale covers $(X', x') \rightarrow (X, x)$ with $x' \in X'(\overline{K})$ a choice of lift of the base point x . This is a spatial diamond, and the canonical projection $\tilde{X} \rightarrow X$ is a pro-finite-étale torsor under the étale fundamental group $\pi_1(X, x)$ of X .

We first note that we have an analogue of Corollary 4.5 in the inverse limit:

Lemma 5.1. *We have $\tilde{\pi}_* \mathcal{O} = \mathcal{O}$ on $\text{Perf}_{K, v}$, and similarly for \mathcal{O}_1^\times , \mathcal{O}^\times , \mathcal{O}^+ and \mathcal{O}^{+a}/p^k .*

Proof. We start with \mathcal{O}^+/p^k : For this we deduce from Proposition 4.12.2 and Corollary 4.5:

$$\mathcal{O}^+/p^k(\tilde{X} \times Y) \stackrel{a}{\cong} \varinjlim \mathcal{O}^+/p^k(X' \times Y) \stackrel{a}{\cong} \mathcal{O}^+/p^k(Y)$$

. The case of \mathcal{O} follows by taking the limit over k and inverting p . The cases of \mathcal{O}^\times , \mathcal{O}^\times and \mathcal{O}_1^\times follow as these are subsheaves. \square

We are finally equipped to prove that the Hodge–Tate sequence for \mathcal{O}_1^\times is short exact:

Proof of Proposition 2.15. We consider the morphism of logarithm long exact sequences

$$\begin{array}{ccccccccc} \pi_*^\diamond \mathcal{O} & \longrightarrow & R^1 \pi_{\tau*}^\diamond \mu_{p^\infty} & \longrightarrow & R^1 \pi_{\tau*}^\diamond \mathcal{O}_1^\times & \longrightarrow & R^1 \pi_{\tau*}^\diamond \mathcal{O} & \longrightarrow & R^2 \pi_{\tau*}^\diamond \mu_{p^\infty} \\ \wr \uparrow & & \wr \uparrow & & \uparrow & & \uparrow & & \wr \uparrow \\ (\pi_* \mathcal{O})^\diamond & \rightarrow & (R^1 \pi_{\text{ét}*} \mu_{p^\infty})^\diamond & \rightarrow & (R^1 \pi_{\text{ét}*} \mathcal{O}_1^\times)^\diamond & \rightarrow & (R^1 \pi_{\text{ét}*} \mathcal{O})^\diamond & \rightarrow & (R^2 \pi_{\text{ét}*} \mu_{p^\infty})^\diamond \end{array}$$

for τ one of ét and v . For either topology, the first vertical arrow is an isomorphism by Corollary 4.5. The second and fifth arrow are isomorphisms by Corollaries 3.19 and 4.8.

For the étale topology, also the fourth arrow is an isomorphism by Proposition 2.6.1, and we conclude the first part by the 5-Lemma.

For the v -topology, by splicing diagram (4) into short exact sequences, we can still deduce from Proposition 2.6.2 that there is a left-exact sequence

$$0 \rightarrow (R^1 \pi_{\text{ét}*} \mathcal{O}_1^\times)^\diamond \rightarrow R^1 \pi_{v*}^\diamond \mathcal{O}_1^\times \xrightarrow{\text{HT log}} H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a.$$

We are left to prove right-exactness. For this it suffices to prove that the map

$$\log : R^1 \pi_{v*}^\diamond \mathcal{O}_1^\times \rightarrow R^1 \pi_{v*}^\diamond \mathcal{O}$$

is surjective. To show this, we first assume that K is algebraically closed, then we can argue like in [14, §3.5] (see the discussion surrounding [14, diagram (10)]): For any affinoid perfectoid Y , consider the pro-finite-étale Galois cover

$$\tilde{X} \times Y \rightarrow X \times Y$$

with group $G := \pi_1(X, x)$. By Lemma 5.1, we have $H_v^0(\tilde{X} \times Y, \mathcal{O}^+) = \mathcal{O}^+(Y)$. The Cartan–Leray sequence thus combines with the logarithm to a commutative diagram:

$$(7) \quad \begin{array}{ccc} \text{Hom}_{\text{cts}}(G, \mathcal{O}_1^\times(Y)) & \rightarrow & H_v^1(X \times Y, \mathcal{O}_1^\times) \\ \downarrow \log & & \downarrow \log \\ \text{Hom}_{\text{cts}}(G, \mathcal{O}(Y)) & \rightarrow & H_v^1(X \times Y, \mathcal{O}). \end{array}$$

We aim to see that the right vertical map becomes surjective upon sheafification in Y . The left morphism becomes surjective since $\log : \mathcal{O}_1^\times \rightarrow \mathcal{O}$ is and since the maximal torsionfree abelian pro- p -quotient of $G = \pi_1(X, x)$ is a finite free \mathbb{Z}_p -module [14, Corollary 3.12]. It thus suffices to see that the bottom map is surjective: By Proposition 4.2.2.(ii) we have $H_v^1(X \times Y, \mathcal{O}) = H_v^1(X, \mathcal{O}) \otimes \mathcal{O}(Y)$. Since $\text{Hom}_{\text{cts}}(G, \mathcal{O}(Y)) = \text{Hom}_{\text{cts}}(G, K) \otimes_K \mathcal{O}(Y)$, this map is $- \otimes_K \mathcal{O}(Y)$ applied to the same map in the case of $Y = \text{Spa}(K)$,

$$\text{Hom}_{\text{cts}}(\pi_1(X, x), K) \rightarrow H_v^1(X, \mathcal{O}),$$

which is an isomorphism since $H^1(\tilde{X}, \mathcal{O}) = 0$ by [14, Proposition 4.9].

Returning to the case of general perfectoid K , consider the inverse system of finite sub-extensions $K \subseteq L \subseteq C$, then there is base-change morphism of logarithm exact sequences

$$\begin{array}{ccccc} H_v^1(X \times Y_L, \mathcal{O}_1^\times) & \xrightarrow{\log} & H_v^1(X \times Y_L, \mathcal{O}) & \xrightarrow{\partial_L} & H_v^2(X \times Y_L, \mu_{p^\infty}) \\ \downarrow & & \downarrow & & \downarrow \\ H_v^1(X \times Y_C, \mathcal{O}_1^\times) & \xrightarrow{\log} & H_v^1(X \times Y_C, \mathcal{O}) & \longrightarrow & H_v^2(X \times Y_C, \mu_{p^\infty}) \end{array}$$

Let x be an element in the middle of the top row, then by the algebraically closed case, the image of x in the bottom row can be lifted along \log after passing to an étale cover of Y_C . Using that $Y_{C, \text{ét-qcqs}} = 2\text{-}\varinjlim Y_{L, \text{ét-qcqs}}$, we can assume that this étale cover comes via pullback from Y_L . Replacing Y_L by this cover, we see that the image of $\partial_L(x)$ in the bottom row vanishes. But the rightmost vertical map becomes an isomorphism in the colimit over L by [32, Proposition 14.9]. Hence $\partial_L(x)$ vanishes for L large enough, and we find the desired lift on the étale cover $Y_L \rightarrow Y$. \square

Finally, we need to see that $(R^2\pi_{\text{ét}*}\mathcal{O}_1^\times)^\diamond \rightarrow R^2\pi_{v*}^\diamond\mathcal{O}_1^\times$ is injective:

Proof of Lemma 2.14.(C). We argue as in the last proof, but in one degree higher: Let

$$\begin{aligned} C_1 &:= \text{coker}(\log: R^1\pi_{\text{ét}*}\mathcal{O}_1^\times \rightarrow R^1\pi_{\text{ét}*}\mathcal{O})^\diamond, \\ C_2 &:= \text{coker}(\log: R^1\pi_{v*}^\diamond\mathcal{O}_1^\times \rightarrow R^1\pi_{v*}^\diamond\mathcal{O}). \end{aligned}$$

By Propositions 2.15 and 2.6, these fit into a commutative diagram

$$\begin{array}{ccccc} R^1\pi_{v*}^\diamond\mathcal{O}_1^\times & \longrightarrow & R^1\pi_{v*}^\diamond\mathcal{O} & \longrightarrow & C_2 \\ \uparrow & & \uparrow & & \uparrow \\ (R^1\pi_{\text{ét}*}\mathcal{O}_1^\times)^\diamond & \longrightarrow & (R^1\pi_{\text{ét}*}\mathcal{O})^\diamond & \longrightarrow & C_1, \end{array}$$

in which the first two vertical maps both have cokernel $\tilde{\Omega}_X^1 \otimes \mathbb{G}_a$. As the second vertical map is injective, this shows that the natural map $C_1 \rightarrow C_2$ is an isomorphism. Continuing the outer diagram in (4) further to the right, these terms fit into a long exact sequence

$$\begin{array}{ccccccccc} 0 & \rightarrow & C_2 & \longrightarrow & R^2\pi_{v*}^\diamond\mu_{p^\infty} & \longrightarrow & R^2\pi_{v*}^\diamond\mathcal{O}_1^\times & \longrightarrow & R^2\pi_{v*}^\diamond\mathcal{O} & \longrightarrow & R^3\pi_{v*}^\diamond\mu_{p^\infty} \\ & & \uparrow \wr & & \uparrow \wr & & \uparrow & & \uparrow & & \uparrow \wr \\ 0 & \rightarrow & C_1 & \rightarrow & (R^2\pi_{\text{ét}*}\mu_{p^\infty})^\diamond & \rightarrow & (R^2\pi_{\text{ét}*}\mathcal{O}_1^\times)^\diamond & \rightarrow & (R^2\pi_{\text{ét}*}\mathcal{O})^\diamond & \rightarrow & (R^3\pi_{\text{ét}*}\mu_{p^\infty})^\diamond \end{array}$$

in which the first, second and last vertical arrows are isomorphisms by Corollary 4.9. The fourth arrow is injective by Proposition 4.4. It follows that the middle arrow is injective. \square

This finishes the proof of Lemmas 2.14 and 2.16. As explained in Section 2.3, this in turn completes the proof parts 1 and 2 of Theorem 2.7.

To get the partial splitting in part 3, we first introduce some notation: Let

$$\mathcal{A} := H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a.$$

We now use that the exponential $\exp: p\mathcal{O}^+ \rightarrow \mathbb{G}_m$ defines a partial inverse to \log on the subspace $1 + p\mathcal{O}^+ \subseteq \mathbb{G}_m$. Moreover, for varying affinoid perfectoid Y , the natural maps $H_v^1(X, \mathcal{O}^+) \otimes_{K^+} \mathcal{O}^+(Y) \rightarrow H_v^1(X \times Y, \mathcal{O}^+)$ induce a morphism

$$H_v^1(X, \mathcal{O}^+) \otimes \mathbb{G}_a^+ \rightarrow R^1\pi_{v*}^\diamond p\mathcal{O}^+.$$

Let $\mathcal{A}^+ \subseteq \mathcal{A}$ be its image under $R^1\pi_{v*}^\diamond p\mathcal{O}^+ \rightarrow R^1\pi_{v*}^\diamond \mathcal{O} \rightarrow \mathcal{A}$, then these combine to a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Pic}_{X,\acute{e}t}^\diamond & \longrightarrow & \mathbf{Pic}_{X,v}^\diamond & \xrightarrow{\text{HT log}} & \mathcal{A} \\ & & \uparrow & & \uparrow^{\text{exp}} & & \uparrow \\ 0 & \longrightarrow & H_{\acute{e}t}^1(X, \mathcal{O}^+) \otimes \mathbb{G}_a^+ & \longrightarrow & H_v^1(X, \mathcal{O}^+) \otimes \mathbb{G}_a^+ & \xrightarrow{\text{HT}^+} & \mathcal{A}^+ \end{array}$$

Since the image of $H_v^1(X, \mathcal{O}^+) \rightarrow H_v^1(X, \mathcal{O})$ is an almost finite free K^+ -module, we can find a splitting $s : H^0(X, \tilde{\Omega}) \rightarrow H_v^1(X, \mathcal{O})$ that induces a splitting of HT^+ , and thus of HT log .

The part of 2 about tangent spaces also follows from the bottom row of the diagram.

This finishes the proof of the Diamantine Picard Comparison Theorem 2.7. \square

As usual, Theorem 2.7 in fact yields a precise description of the Picard group:

Corollary 5.2. *Let Y be a perfectoid space over K and assume that the rigid Picard functor $\mathbf{Pic}_{X,\acute{e}t}$ is represented by an adic space G . Then any $x \in X(K)$ defines an isomorphism*

$$\mathbf{Pic}_{\acute{e}t}(X \times Y) = \mathbf{Pic}_{\acute{e}t}(Y) \times G(Y).$$

Proof. This follows from Theorem 2.7 and Lemma 4.13 which applies by Corollary 4.5. \square

Proof of Corollary 2.9. 1. By Theorem 2.7.1-2, the sheaf $(\mathbf{Pic}_{X,\acute{e}t})^\diamond$ is the kernel of a morphism of v -sheaves on Perf_K , hence it is itself a v -sheaf.

2. Let us for simplicity write $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ for the Picard functor defined on all of $\text{LSD}_{K,\acute{e}t}$, and similarly for the v -topology. We claim that for rigid Y , the natural sequence

$$1 \rightarrow \mathbf{Pic}_{X,\acute{e}t}^\diamond(Y) \rightarrow \mathbf{Pic}_{X,v}^\diamond(Y) \xrightarrow{\text{HT log}} H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathcal{O}(Y)$$

is still left-exact. It then follows that $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ is still the kernel of HT log on rigid spaces. But HT log is a morphism of v -sheaves, and thus its kernel is a v -sheaf.

To see that the sequence is left-exact, we study the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & H_{\acute{e}t}^1(Y, \mathcal{O}^\times) & \longrightarrow & H_{\acute{e}t}^1(X \times Y, \mathcal{O}^\times) & \longrightarrow & \mathbf{Pic}_{X,\acute{e}t}^\diamond(Y) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & H_v^1(Y, \mathcal{O}^\times) & \longrightarrow & H_v^1(X \times Y, \mathcal{O}^\times) & \longrightarrow & \mathbf{Pic}_{X,v}^\diamond(Y) \longrightarrow 1 \\ & & \downarrow^{\text{HT log}} & & \downarrow^{\text{HT log}} & & \downarrow^{\text{HT log}} \\ 0 & \rightarrow & H^0(Y, \tilde{\Omega}_Y^1) & \rightarrow & H^0(X \times Y, \tilde{\Omega}_{X \times Y}^1) & \rightarrow & H^0(X, \tilde{\Omega}_X^1) \otimes \mathcal{O}(Y) \rightarrow 0 \end{array}$$

Here the first two columns are the left exact Hodge–Tate logarithm sequences [14, Theorem 1.3] associated to the rigid spaces Y and $X \times Y$, respectively. The first two rows are the exact sequences from Lemma 4.13. The bottom row is also exact, since

$$\Omega^1(X \times Y) = (\Omega^1(Y) \otimes_K \mathcal{O}(X)) \oplus (\Omega^1(X) \otimes_K \mathcal{O}(Y))$$

and $\mathcal{O}(X) = K$. The leftmost HT log in the diagram becomes surjective after étale sheafification in Y . It follows from a diagram chase that also the third column is exact.

3. Clear from Theorem 2.7.1.

4. By part 2, if $Y' \rightarrow Y$ is a pro-étale perfectoid cover of a rigid space, we have descent for étale line bundles along $X \times Y' \rightarrow X \times Y$. One also has v -descent for maps into G .

5. When $\mathbf{Pic}_{X,v}^\diamond$ is representable by a rigid group, then the short exact sequence (3) in Theorem 2.7.2 expresses $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ as the kernel of a morphism of rigid groups, hence $\mathbf{Pic}_{X,\acute{e}t}^\diamond$ is itself represented by a rigid group.

To see the converse, let \mathcal{A}^+ be the bounded open subgroup of $\mathcal{A} = H^0(X, \tilde{\Omega}_X^1) \otimes_K \mathbb{G}_a$ described in Theorem 2.7.3 and consider for any $n \in \mathbb{N}$ the short exact sequence

$$0 \rightarrow \mathbf{Pic}_{X,\acute{e}t} \rightarrow \mathbf{Pic}_{X,v}^{(n)} \rightarrow p^{-n}\mathcal{A}^+ \rightarrow 0$$

defined by the fibre of (3) over the open subgroup $p^{-n}\mathcal{A}^+$. Then we have

$$\mathbf{Pic}_{X,v} = \bigcup_{n \in \mathbb{N}} \mathbf{Pic}_{X,v}^{(n)},$$

so it suffices to prove that each $\mathbf{Pic}_{X,v}^{(n)}$ is represented by a rigid space. For any n we have a morphism of short exact sequences of v -sheaves, exact in the étale topology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Pic}_{X,\acute{e}t} & \longrightarrow & \mathbf{Pic}_{X,v}^{(n)} & \longrightarrow & p^{-n}\mathcal{A}^+ \longrightarrow 0 \\ & & \downarrow [p^n] & & \downarrow [p^n] & & \wr \downarrow \cdot p^n \\ 0 & \longrightarrow & \mathbf{Pic}_{X,\acute{e}t} & \longrightarrow & \mathbf{Pic}_{X,v}^{(0)} & \longrightarrow & \mathcal{A}^+ \longrightarrow 0. \end{array}$$

By Theorem 2.7.3, the bottom sequence is split. In particular, the middle term is represented by a (smooth) rigid group variety if and only if the first term is.

By [32, Lemma 15.6] any diamond that is étale over a rigid space comes from a rigid space, so it suffices to prove that the middle arrow is an étale morphism of diamonds.

To prove this, we may work v -locally and assume that K is algebraically closed. By the long exact sequence of $\pi_{\acute{e}t,*}^{\diamond}$, the kernel of the left vertical map is $R^1\pi_{\acute{e}t,*}^{\diamond}\mu_{p^n}$ which by Corollary 4.8 is represented by the finite étale rigid group $G_n := H_{\acute{e}t}^1(X, \mu_{p^n})$. It follows from the diagram that then also the middle vertical map is an étale torsor under G_n . \square

5.1. Translation-invariant Picard functors. If $X = A$ is a proper rigid group variety, i.e. an abeloid variety, then there is a variant of the Picard functor that is frequently used, for example by Bosch–Lütkebohmert [3, §6]: the translation-invariant Picard functor. We finish this section by noting that the Diamantine Picard Comparison Theorem easily implies a translation-invariant version. We will use this in [11] to prove a uniformisation result for abeloids.

Definition 5.3. Let A be a connected smooth proper rigid group. Denote by $\pi_1, \pi_2, m : A \times A \rightarrow A$ the two projection maps and the group operation, respectively. For any rigid or perfectoid space Y , we denote by $\mathbf{Pic}_{\acute{e}t}^{\tau}(A \times Y)$ the kernel of the map

$$\pi_1^* + \pi_2^* - m^* : \mathbf{Pic}_{\acute{e}t}(A \times Y) \rightarrow \mathbf{Pic}_{\acute{e}t}(A \times A \times Y).$$

The translation-invariant Picard functor \mathbf{Pic}_A^{τ} of A is defined as the kernel of the morphism

$$\pi_1^* + \pi_2^* - m^* : \mathbf{Pic}_A \rightarrow \mathbf{Pic}_{A \times A}.$$

We analogously define the translation invariant diamantine Picard functor $\mathbf{Pic}_{A,\acute{e}t}^{\diamond\tau} \subseteq \mathbf{Pic}_{A,\acute{e}t}^{\diamond}$.

By duality theory of abeloids, developed by Bosch–Lütkebohmert [3, §6], the functor \mathbf{Pic}_A^{τ} is represented by an abeloid variety A^{\vee} that is called the dual abeloid. We deduce:

Corollary 5.4. *We have $\mathbf{Pic}_{A,\acute{e}t}^{\diamond\tau} = (\mathbf{Pic}_A^{\tau})^{\diamond}$ and this functor is represented by $A^{\vee\diamond}$. In particular, for any perfectoid space Y over K , we have $\mathbf{Pic}_{\acute{e}t}^{\tau}(A \times Y) = \mathbf{Pic}(Y) \times A^{\vee}(Y)$.*

Proof. The first statement follows from Theorem 2.7.1 by exactness of $-\diamond$. The last part follows from Lemma 4.13 which shows that specialisation at $0 \in A(K)$ defines an isomorphism $\mathbf{Pic}_{\acute{e}t}(A \times Y) = \mathbf{Pic}(Y) \times \mathbf{Pic}_A^{\diamond}(Y)$. The same holds for A replaced by $A \times A$. We get the desired statement by comparing kernels on both sides of the maps in Definition 5.3. \square

APPENDIX A. LEMMAS ON COMPLEXES OF BANACH ALGEBRAS

Let R be any ring and let $\varpi \in R$ be any element. We recall that an R -module M is said to have bounded ϖ -torsion if $M[\varpi^\infty] := \cup_{k \in \mathbb{N}} M[\varpi^k]$ is equal to $M[\varpi^n]$ for some $n \in \mathbb{N}$.

Lemma A.1. *If $A \rightarrow B \rightarrow C \rightarrow D$ is an exact sequence of R -modules in which A and D are killed by ϖ^n for some $n \in \mathbb{N}$ and B has bounded ϖ -torsion, then C has bounded ϖ -torsion.*

Proof. This is an elementary diagram chase: Choose n large enough such that $B[\varpi^\infty] = B[\varpi^n]$. We claim that $C[\varpi^\infty] = C[\varpi^{2n}]$. Let $x \in C[\varpi^N]$ for some N . Then $\varpi^n x$ lifts to an element y of B , and $\varpi^N y$ goes to $\varpi^{N+n} x = 0$ in C , hence lifts to A . Since A is killed by ϖ^n , this implies $\varpi^{N+n} y = 0$. Since $B[\varpi^\infty] = B[\varpi^n]$, this shows $\varpi^n y = 0$, hence $\varpi^{2n} x = 0$. \square

Lemma A.2. *Let $C_1^\bullet \rightarrow C_2^\bullet \rightarrow C_3^\bullet$ be a short exact sequence of complexes of R -modules. Suppose that for each $n \in \mathbb{Z}$, the module $H^n(C_3^\bullet)$ has bounded ϖ -torsion and $H^n(C_2^\bullet)$ is killed by ϖ^k for some $k \in \mathbb{N}$. Then $H^n(C_1^\bullet)$ has bounded ϖ -torsion.*

Proof. We apply Lemma A.1 to the long exact sequence of cohomologies. \square

Let (K, K^+) be any non-archimedean field. We now specialise to the setting that $R = K^+$ and $\varpi \in K^+$ is a pseudo-uniformiser.

Lemma A.3. *Let C^\bullet be a bounded complex of ϖ -torsionfree ϖ -adically complete K^+ -modules. Suppose that $C^\bullet[\frac{1}{\varpi}]$ is a complex of K -Banach modules such that $H^n(C^\bullet[\frac{1}{\varpi}])$ is finite-dimensional for all $n \in \mathbb{Z}$. Then:*

1. $H^n(C^\bullet)$ has bounded ϖ -torsion.
2. For any ϖ -torsionfree K^+ -module S , we have

$$H^n(C^\bullet \hat{\otimes}_{K^+} S) = H^n(C^\bullet) \hat{\otimes}_{K^+} S$$

where $\hat{\otimes}$ denotes the ϖ -adically completed tensor product.

Proof. By [27, §II.5 Lemma 1], there exists a perfect complex P^\bullet of K -vector spaces and a quasi-isomorphism $f : P^\bullet \rightarrow C^\bullet[\frac{1}{\varpi}]$. Choosing K^+ -lattices in each P^n and rescaling if necessary, we can find a perfect complex $P^{+, \bullet}$ of K^+ -modules such that $P^{+, \bullet}[\frac{1}{\varpi}] = P^\bullet$ and such that f admits a K^+ -model

$$f^+ : P^{+, \bullet} \rightarrow C^\bullet.$$

Let L be the mapping cone of f^+ , then we have a short exact sequence of K^+ -complexes

$$0 \rightarrow C^\bullet \rightarrow L \rightarrow P^{+, \bullet}[1] \rightarrow 0.$$

Since f is a quasi-isomorphism, $L[\frac{1}{\varpi}]$ is an exact complex of K -Banach modules, so we have $H^n(L)[\frac{1}{\varpi}] = 0$. By a standard argument, it now follows from Banach's Open Mapping Theorem that $H^n(L)$ has bounded ϖ -torsion for all $n \in \mathbb{N}$. (We learnt this argument from the proof of [29, Proposition 6.10]. See e.g. [16, Lemma A.3.1] for a proof of the statement.)

The modules $H^n(P^{+, \bullet})$ are finitely presented K^+ -modules, hence they also have bounded ϖ -torsion. Therefore, Lemma A.2 applies and shows part 1.

For part 2, we note that S is a flat K^+ -module. The result therefore follows from part 1 by [16, Lemma A.3.6], or alternatively [17, Proposition A.3.1]. \square

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