PICARD FUNCTORS OF PRO-ÉTALE UNIVERSAL COVERS

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Abstract. We study Picard functors of the universal pro-finite-étale perfectoid cover $\widetilde{X}$ of a smooth rigid space X in three different setups: First, we treat the case when X is a smooth projective curve. We show that for such X, the Picard functor of $\widetilde{X}$ remembers the Jacobians of the reduction of all finite étale covers of X, consistent with conjectures on pro-étale uniformisation. Second, we treat abelian varieties: By way of a perfectoid Appell–Humbert Theorem, we show that the Néron–Severi group of $\widetilde{A}$ is a finite $\mathbb{Q}$-vector space which can have strictly larger dimension than the Néron-Severi rank of A. Nevertheless, every line bundle on $\widetilde{A}$ comes from some abeloid variety. Third, we consider the case of perfectoid tori, which we prove to have very large Picard groups, in contrast to rigid tori.

1. Introduction

Let K be a complete algebraically closed non-archimedean field over $\mathbb{Z}_p$. Let X be a connected smooth rigid space over K considered as an adic space and fix a base-point $x \in X(K)$. We consider the pro-finite-étale universal cover of X, defined as the cofiltered limit in the category of diamonds of all connected finite étale covers of X with a lift of x to $X'(K)$:

$$\widetilde{X} = \varprojlim_{(X',x')\to(X,x)} X'.$$

In [17, §4], we have argued that this is an analogue of the universal cover of complex manifolds in complex geometry: First it is "simply connected" in the weak sense that any finite étale cover of $\widetilde{X}$ is split and $H^1(\widetilde{X},\mathcal{O}) = 0$. Second, it is uniformising in the sense that $\widetilde{X}$ becomes isomorphic for a large class of different X in the case when X is an abelian variety [16], and conjecturally also when X is a curve. Third, we had shown in [17][14] that $\widetilde{X}$ can be used to replace the role of the complex universal cover in the p-adic Simpson correspondence. Each of these features naturally leads to the following natural question:

Question 1.1. What is $\mathrm{Pic}_{\mathrm{an}}(\widetilde{X})$? More precisely, consider the natural map

$$\varinjlim_{i\in I} \mathrm{Pic}_{\mathrm{an}}(X) \to \mathrm{Pic}_{\mathrm{an}}(\widetilde{X}). \qquad (1)$$

What is its kernel, and how far is it from being surjective?

In this article, we answer Question 1.1 when X is a curve, or an abelian variety, or a torus. In fact, we will describe not only the Picard groups, but even the Picard functors. But let us first state a particularly clean special case of our main result in terms of Picard groups:

Theorem 1.2. *(1) If X is a curve over $\mathbb{C}_p$ or $\mathbb{C}_p^\flat$, then the degree defines an isomorphism*

$$\mathrm{Pic}(\widetilde{X}) \cong \mathbb{Q}.$$

(2) If X is an abeloid variety over $\mathbb{C}_p$ or $\mathbb{C}_p^\flat$, then $\mathrm{Pic}(\widetilde{X})$ is the finite $\mathbb{Q}$-vector space

$$\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Hom}(\widetilde{X}, \widetilde{X}^\vee)^{\mathrm{sym}}$$

of symmetric homomorphisms from $\widetilde{X}$ to the universal cover of the dual abeloid variety $X^\vee$. Not every line bundle L on $\widetilde{X}$ comes from a finite cover of X, but we can always find some abeloid variety X' such that L is the pullback along some map $\widetilde{X} \to X'$.

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1.1. **Motivation 1: The p-adic Simpson correspondence and a question of Faltings.** Our main motivation for Question 1.1 stems from an open question of Faltings in the context of p-adic non-abelian Hodge theory: When X is a smooth projective curve, Faltings has constructed in his influential work [8] a fully faithful p-adic Simpson functor

$$S : \begin{Bmatrix}\text{cts. representations of } \pi_1(X,x)\\ \text{on fin. dim. } K\text{-vector spaces}\end{Bmatrix} \hookrightarrow \Big\{\text{Higgs bundles on } X\Big\}.$$

Such a functor was more generally constructed in [19] for any connected smooth proper rigid space, and the universal cover $\widetilde{X}$ plays a key role in the construction. Faltings asks in his article how to describe the essential image of S, and this is now the main open question about the p-adic Simpson correspondence. Partial answers for curves are known due to Deninger–Werner [7], but especially following the recent work of Andreatta [1], Faltings' question is now completely open and is expected to be very hard.

By construction of the p-adic Simpson functor, Faltings' question can be rephrased as asking which vector bundles on X become trivial after pullback along $\widetilde{X} \to X$. Based on this reformulation, an answer to Faltings' question is known in the case of line bundles: In [17][14], we described the essential image of the characters $\pi_1(X,x) \to K^\times$ under the p-adic Simpson functor. They key idea is to consider moduli spaces of both sides and to study the kernel of the pullback map between moduli functors $\mathbf{Pic}_X \to \mathbf{Pic}_{\widetilde{X}}$. Namely, we showed that this behaves roughly like a reduction map from an abelian variety to its special fibre. However, we only described the kernel, stopping short of describing $\mathbf{Pic}_{\widetilde{X}}$ itself.

Since this is the only known method to describe the essential image of the p-adic Simpson functor for line bundles, this strongly suggests that in order to answer Faltings' question in higher rank $n \in \mathbb{N}$, we should study the geometry of the moduli stack $\mathrm{Bun}_{\widetilde{X},n}$ of rank n vector bundles on $\widetilde{X}$. Recall that classically, the description of the moduli space of semi-stable bundles on a curve relies on describing vector bundles of degree 0 as successive extensions of line bundles of varying degree. This requires an understanding of the geometry of the full Picard variety (see e.g. [29]). We expect the same to be true for the universal cover $\widetilde{X}$. For this reason, in order to understand the pullback map $\mathrm{Bun}_{X,n} \to \mathrm{Bun}_{\widetilde{X},n}$ for $n \geq 2$, we want to first understand all of $\mathbf{Pic}_{\widetilde{X}}$, not just the kernel of the pullback map.

In summary, we propose that the first step in answering Faltings' question to understand the full Picard functor $\mathbf{Pic}_{\widetilde{X}}$. This is the goal of this article.

1.2. **Motivation 2: Uniformisation.** In [16], we had seen that universal covers $\widetilde{A}$ of abelian varieties A over K are "locally constant" in the moduli space, and we asked whether the same holds for curves. We even asked if a stronger result could be true, namely that all curves C of genus ≥ 2 might have isomorphic universal covers $\widetilde{C}$, like in complex geometry.

A natural test for this conjecture is to compare invariants of the covers $\widetilde{C}$ as C varies, in particular cohomology with various coefficients. However, due to the "pro-étale simply connectedness" of $\widetilde{C}$, the étale cohomology of $\widetilde{C}$ with $\mathbb{Z}_p$- or $\mathbb{Z}_l$-coefficients vanishes, as does its cohomology with $\mathcal{O}$ or $\mathcal{O}^+$-coefficients.

In the case of abelian varieties, we had seen that instead, Picard groups, i.e. cohomology with $\mathcal{O}^\times$-coefficients, provide a useful invariant to study morphisms between universal covers. While these turn out to be yet too coarse for curves, we show in this article that the Picard *functor* has enough structure to show that certain universal covers cannot be isomorphic.

In cases in which the universal cover is locally constant in p-adic families, e.g. for abeloids, studying line bundles of $\widetilde{X}$ means studying the behaviour of Picard ranks in families.

In the case of abeloids, the category of vector bundles on $\widetilde{A}$ is moreover of interest for a generalisation to abeloids of the Theorem of Matsushima, Morimoto, Miyanishi and Mukai [27, Theorem 4.17] characterising homogeneous vector bundles on A.

1.3. **Main results: Picard functors.** In this article, we answer Question 1.1 in three new settings which we now describe. Each of them sheds some new light on the situation; for example, in two of these cases, the map (1) is not surjective, each for a different reason.

The three instances of Question 1.1 that we study are

(a) when X is a smooth projective curve of genus ≥ 1,
(b) when X is an abeloid, or
(c) when X is a rigid torus.

In all of these cases, $\widetilde{X}$ is perfectoid.

Let X be any smooth proper rigid space over K of characteristic 0. The (smooth) rigid analytic Picard functor is defined by

$$\mathbf{Pic}_X : \{\text{Smooth rigid spaces over } K\} \to \text{Sets}, \quad S \mapsto \operatorname{Pic}(S \times X)/\operatorname{Pic}(S).$$

Conjecturally, this is always represented by a rigid group variety, and this is known for example for curves and abeloids. If that is the case, the identity component of $\mathbf{Pic}_X$, which is conjectured to always be semi-abeloid, is denoted by $\mathbf{Pic}^0_X$, and the quotient

$$\operatorname{NS}(X) := \mathbf{Pic}_X/\mathbf{Pic}^0_X$$

is called the Néron–Severi group, which is conjecturally a finitely generated abelian group.

We had shown in [14] that when $\mathbf{Pic}_X$ is representable, the diamantine Picard functor

$$\mathbf{Pic}^{\diamondsuit}_X : \{\text{Perfectoid spaces over } K\} \to \text{Sets}, \quad S \mapsto \operatorname{Pic}(S \times X)/\operatorname{Pic}(S)$$

is represented by the same rigid space as $\mathbf{Pic}_X$. We can analogously define a "diamantine Picard functor" $\mathbf{Pic}_{\widetilde{X}}$ for the universal cover. Our first goal in this article is to show that there is again a natural notion of $\mathbf{Pic}^0$ and Néron–Severi groups in this context:

Theorem 1.3. *Let X be a smooth proper rigid space over an algebraically closed non-archimedean field K of characteristic 0. Then we have a short exact sequence*

$$0 \to \varinjlim_{X' \to X} \mathbf{Pic}^{\mathrm{tt}}_X \to \varinjlim_{X' \to X} \mathbf{Pic}_X \to \mathbf{Pic}_{\widetilde{X}} \to \underline{Q} \to 0$$

where Q is the locally constant sheaf associated to the discrete abelian group

$$Q := \operatorname{Pic}(\widetilde{X})/ \varinjlim_{X' \to X} \operatorname{Pic}(X).$$

In particular, we get natural definitions

$$\mathbf{Pic}^0_{\widetilde{X}} := \varinjlim \operatorname{im}(\mathbf{Pic}^0_{X'} \to \mathbf{Pic}_{\widetilde{X}}),$$

$$\operatorname{NS}(\widetilde{X}) := \varinjlim \operatorname{coker}(\operatorname{Pic}^0(X') \to \operatorname{Pic}(\widetilde{X})).$$

This reduces the study of the Picard *functor* of $\widetilde{X}$ to that of the Picard *group* $\operatorname{Pic}(\widetilde{X})$ and that of the Picard functor of each rigid space X'.

1.4. **Main results – Curves.** Let now $X = C$ be a connected smooth proper curve of genus ≥ 1 over K, which we assume to be of characteristic 0. Recall that in the rigid setting, the rigid analytic Picard functor

$$\mathbf{Pic}_C : \{\text{Rigid spaces over } K\} \to \text{Sets}, \quad S \mapsto \operatorname{Pic}(S \times C)/\operatorname{Pic}(S).$$

sits in a short exact sequence of rigid group varieties

$$0 \to J_C \to \mathbf{Pic}_C \xrightarrow{\deg} \underline{\mathbb{Z}} \to 0,$$

where J_C is the Jacobian of C.

Theorem 1.4. *The degree map induces an exact sequence*

$$0 \to \varinjlim_{C' \to C} J_{C'}/J^{\mathrm{tt}}_{C'} \to \mathbf{Pic}_{\widetilde{C}} \xrightarrow{\deg} \underline{\mathbb{Q}} \to 0$$

where $C' \to C$ ranges over all finite étale covers and where $J_{C'}$ is the Jacobian of C'. In particular, if $K = \mathbb{C}_p$, there is a natural isomorphism

$$\deg : \operatorname{Pic}(\widetilde{C}) \xrightarrow{\sim} \mathbb{Q}.$$

When C has good reduction, this shows that the Picard functor "remembers" the Jacobians of all finite étale cover of the reduction of C. We can use this to prove:

Corollary 1.5. *Let K be either of characteristic p or an extension of $\mathbb{C}_p$ with residue field strictly larger than $\overline{\mathbb{F}}_p$. Then there are curves C_1 and C_2 over K such that*

$$\widetilde{C}_1 \neq \widetilde{C}_2.$$

In other words, the strong p-adic Uniformization Conjecture only has a chance of being true over $\mathbb{C}_p$ (or extensions that don't increase the residue field).

1.5. **Main results – Abeloids.** Let A be an abeloid variety, i.e. a connected smooth proper rigid group variety, for example an abelian variety. Using a perfectoid version of the classical Appell–Humbert Theorem, we then obtain a description of the Picard group first of the universal cover $\widetilde{A}$:

Theorem 1.6. *There is a natural exact sequence*

$$0 \to A^\vee/A^{\vee\mathrm{tt}} \to \mathrm{Pic}(\widetilde{A}) \to \underline{\mathrm{Hom}(\widetilde{A}, \widetilde{A}^\vee)}^{\mathrm{sym}} \to 0.$$

Here the first term is $\mathrm{Pic}^0(\widetilde{A})$ and the third is $\mathrm{NS}(\widetilde{A})$, which is a finite dimensional $\mathbb{Q}$-vector space. The first term vanishes if $K = \mathbb{C}_p$.

In particular, the Néron–Severi group of $\widetilde{A}$ can be described in a very similar way as this can be done for complex abelian varieties following Riemann, or in the rigid setting following Bosch–Lütkebohmert. We can use this to get an interesting take on Question 1.1: While the theorem shows that in general, the map

$$\mathrm{Pic}(A) \otimes \mathbb{Q} \to \mathrm{Pic}(\widetilde{A})$$

may have non-trivial cokernel, the second part can be used to show that $\mathrm{Pic}(\widetilde{A})$ can be fully described in terms of abeloids when we take into account the fact that there are many different abeloids with isomorphic covers [16]:

Corollary 1.7. *For every line bundle $L \in \mathrm{Pic}(\widetilde{A})$, there is an abeloid variety A' and an isomorphism $\widetilde{\widetilde{A}} \cong \widetilde{A}'$ such that L is in the image of $\mathrm{Pic}(A') \to \mathrm{Pic}(\widetilde{A})$.*

This gives a complete answer to Question 1.1 in the case of abeloid varieties.

1.6. **The non-quasicompact case.** In [17], we have compared the analytic Picard group $\mathrm{Pic}_{\mathrm{an}}(X)$ to the v-Picard group $\mathrm{Pic}_v(X^\diamondsuit)$ of the associated diamond, and we have seen that the latter is typically strictly larger. The reason for this discrepancy is that there can be line bundles on perfectoid pro-étale covers $X_\infty \to X$ with descent data for which analytic descent is not effective. This happens for example when X is a torus (as a qualitative statement, this follows from [17, Theorem 1.2b]). In this regard, this paper complements our earlier results by describing the origin of these additional line bundles.

Finally, to complement our study of diamantine Picard groups of proper smooth adic spaces, we also study Question 1.1 in the case of the rigid torus T with its p-adic universal cover

$$\widetilde{T} = \varprojlim_{[p]} T,$$

which equals T^{perf} if $\mathrm{char}\, K = p$. Whilst in the rigid analytic case we have

$$\mathrm{Pic}_{\mathrm{an}}(T) = 1,$$

it turns out that the perfectoid torus $\widetilde{T}$ has huge Picard group. This is related to the failure of sums of the form $\sum_{n\in\mathbb{N}} p^{1/p^n} X^{1/p^n}$ to converge p-adically, giving rise to many glueing data for trivial line bundles on increasing annuli that cannot be trivialised globally (a similar phenomenon appears in the rigid setting for open disks over base fields that are not spherically complete [22, Proposition 6], but not for rigid tori).

In contrast to the cases of curves and abeloids, where the additional v-topological line bundles on X came from descent data on the trivial line bundle, this shows that "most" v-line bundles on T arise from non-trivial line bundles on $\widetilde{T}$.

Notation. Throughout we denote by K a complete algebraically closed non-archimedean field of residue characteristic p. Let $\mathcal{O}_K$ be the ring of integers, $\mathfrak{m}$ the maximal ideal, and Γ the value group. We fix a pseudo-uniformiser $\pi \in \mathfrak{m}$. The group $K^\times/(1+\mathfrak{m})$ will play an important role, and we will sometimes abbreviate it by $K^\times_{/\mathfrak{m}}$ to ease notation.

For topological spaces T, S we write $\mathcal{C}(T,S)$ for the set of continuous morphisms $T \to S$. For any topological group G, we denote by $\widehat{G}$ the subgroup of topologically p-torsion elements

$$\widehat{G} = \{x \in G | x^{p^n} \to 0 \text{ for } n \to \infty\}.$$

We denote by G^{tt} the subgroup of $\widehat{G}$ of topological torsion elements, i.e.

$$G^{\mathrm{tt}} = \{x \in G | x^{n!} \to 0 \text{ for } n \to \infty\}.$$

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2. Picard functor of universal covers

Our first goal is a general result about Picard functors of universal covers:

In this section, let X be a connected smooth proper rigid space over an algebraically closed non-archimedean field extension K of $\mathbb{Q}_p$. We recall from [17, §4.3] the definition of the universal cover of X:

Definition 2.1. Fix a base point $x \in X(K)$. The pro-finite-étale universal cover of X is the diamond

$$\widetilde{X} = \varprojlim_{X' \to X} X'$$

where the limit is over all connected finite étale covers $X' \to X$ together with a lift $x' \in X'(K)$ of x.

One of the main results of [14] was the following result about diamantine Picard functors of universal covers:

Theorem 2.2 ([14, Theorem 4.1])**.** *There is a short exact sequence*

$$0 \to \mathbf{Pic}^{\mathrm{tt}}_X \to \mathbf{Pic}_X \to \mathbf{Pic}_{\widetilde{X}}.$$

Here we the first term denotes the topological torsion subgroup:

Definition 2.3 ([18, §2.2, Proposition 2.14])**.** Let G be any topological group. Then we denote by $G^{\mathrm{tt}} \subseteq G$ the subset of elements $x \in G$ such that $x^{N!} \to 1$ for $N \to \infty$. We denote by $\widehat{G} \subseteq G$ the subset of $x \in G$ such that $x^{p^n} \to 1$ for $n \to \infty$.

Let H be any rigid group over K. Then we denote by $H^{\mathrm{tt}} \subseteq H$ the open subgroup that represents the evaluation functor $\mathbf{Hom}(\widehat{\mathbb{Z}}, H) \to H$. We similarly denote by $\widehat{H} \subseteq H$ the open subgroup represented by $\mathrm{Hom}(\mathbb{Z}_p, H)$, then $\widehat{H}(K) = \widehat{H(K)}$.

The first goal of this section is to complement this result by describing the cokernel on the right. This is achieved by the following result:

Theorem 2.4. *Let X be a smooth proper rigid space over an algebraically closed non-archimedean field K of characteristic 0. Then we have a short exact sequence*

$$0 \to \varinjlim_{X' \to X} \mathbf{Pic}^{\mathrm{tt}}_X \to \varinjlim_{X' \to X} \mathbf{Pic}_X \to \mathbf{Pic}_{\widetilde{X}} \to \underline{Q} \to 0$$

where Q is the locally constant sheaf associated to the discrete abelian group

$$Q := \mathrm{Pic}(\widetilde{X})/\varinjlim_{X'\to X}\mathrm{Pic}(X).$$

Proof. It suffices to prove the following statement:

Proposition 2.5. *Let T be affinoid perfectoid. Then the map*

$$\varinjlim_{X'\to X}\mathrm{Pic}(X'\times T)\times\mathrm{Pic}(\widetilde{X})\to\mathrm{Pic}(\widetilde{X}\times T)$$

is surjective.

As a first step, we claim that the following morphism is surjective:

$$H^1(X\times T,\mathcal{O}^\times)[\tfrac{1}{p}]\times H^1(X,\overline{\mathcal{O}}^\times)\to H^1(X\times T,\overline{\mathcal{O}}^\times).$$

By rigid approximation, we then have

$$H^1(X\times T,\overline{\mathcal{O}}^\times)=\varinjlim H^1(X'\times T',\overline{\mathcal{O}}^\times)$$

where $T\to T'$ ranges through all morphisms to affinoid smooth rigid spaces. We can therefore reduce to showing that for T a connected reduced rigid space, the map

$$H^1(X\times T,\mathcal{O}^\times)[\tfrac{1}{p}]\times H^1(X,\overline{\mathcal{O}}^\times)\to H^1(X\times T,\overline{\mathcal{O}}^\times)\tag{2}$$

is surjective. For this we compare the boundary maps of the exponential sequence for X and $X\times T$, which fit into a morphism of long exact sequences

$$\begin{array}{ccccccc}
 & & H^1(X,\overline{\mathcal{O}}^\times) & \longrightarrow & H^2(X,\mathcal{O}) & \longrightarrow & H^2(X,\mathcal{O}^\times)[\tfrac{1}{p}]\\
 & & \downarrow & \nearrow & \downarrow & & \downarrow\\
H^1(X\times T,\mathcal{O}^\times)[\tfrac{1}{p}] & \longrightarrow & H^1(X\times T,\overline{\mathcal{O}}^\times) & \longrightarrow & H^2(X\times T,\mathcal{O}) & \longrightarrow & H^2(X\times T,\mathcal{O}^\times[\tfrac{1}{p}]).
\end{array}$$

By [15, Lemma 5.11], the bottom boundary map admits a factorisation via the dashed arrow. Since T is a rigid space, it always has a K-point. Specialisation at this point defines a section of the vertical maps, which are thus injective.

Let now α be any class in $H^1_{\text{ét}}(X\times T,\overline{\mathcal{O}}^\times)$. Then chasing the diagram we see that there is a class α' in $H^1_{\text{ét}}(X,\overline{\mathcal{O}}^\times)$ which has the same image in $H^2_{\text{ét}}(X\times T,\mathcal{O})$. Thus $\alpha-\alpha'$ is in the image of $H^1(X\times T,\mathcal{O}^\times)[\frac{1}{p}]$. This proves that the map in (2) is surjective.

We now consider an analogous diagram for $\widetilde{X}$ and an affinoid perfectoid space T: Here we have a morphism of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(\widetilde{X}_T,\mathcal{O}^\times) & \longrightarrow & H^1(\widetilde{X}_T,\overline{\mathcal{O}}^\times) & \longrightarrow & H^2(\widetilde{X}_T,\mathcal{O})\\
 & \uparrow & & \uparrow & & \uparrow\\
 & H^1(X_T,\mathcal{O}^\times[\tfrac{1}{p}])\times\mathrm{Pic}(\widetilde{X}) & \to & H^1(X_T,\overline{\mathcal{O}}^\times)\times H^1(\widetilde{X},\overline{\mathcal{O}}^\times) & \to & H^2(X_T,\mathcal{O})\times H^2(\widetilde{X},\mathcal{O}).
\end{array}$$

It now follows from a diagram chase that the leftmost map is surjective: Let α be an element on the top left and consider its image in the middle. Then by the first part of the proof, this is the preimage of an element $\beta=(\beta_1,\beta_2)$ in the bottom middle term, where the first entry already comes from the left. In particular, β goes to $(0,\delta(\beta_2))$ under the boundary map. The image of this in the top right is 0 as α comes from the top left. But since the rightmost map is injective on the second factor, it follows that β already comes from the bottom left. As the top left morphism is injective, it follows that α is in the image of the leftmost map. □

3. Picard groups of universal covers of curves

Building on the last section, we now study the Picard groups of universal covers of curves: More precisely, let C be a connected smooth proper curve over K. Let A be the Jacobian. Choose a base point $x \in C$, this induces a map $C \to A$. Let g be the genus of C. For a uniform treatment, we explicitly allow the case of $g = 0$, in which case we set $A = 1$.

3.1. The universal cover.

Consider the pro-finite-étale universal cover

$$\widetilde{C} := \varprojlim_{C' \to C} C'$$

given as the limit over all connected finite étale covers $C' \to C$ with a chosen lift of base point. If $g \geq 1$, this is represented by a perfectoid space by [3, Corollary 5.7]. If $g = 0$, we simply have $\widetilde{C} = C$. In either case, in dimension 1, we have the following strengthening of [17, Proposition 3.10].

Proposition 3.1. *We have for any $i \in \mathbb{Z}_{\geq 0}$ and any $N \in \mathbb{N}$:*

$$H^i_v(\widetilde{C}, \mathbb{Z}/N\mathbb{Z}) = H^i_{\text{qproét}}(\widetilde{C}, \mathbb{Z}/N\mathbb{Z}) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

$$H^i_v(\widetilde{C}, \mathcal{O}^+) = H^i_{\text{an}}(\widetilde{C}, \mathcal{O}^+) = \begin{cases} \mathcal{O}_K & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

Proof. For $i = 0$ and $i = 1$, this is proved in [17, Proposition 3.10]. To see the statement for $i > 1$, recall that

$$H^i_v(\widetilde{C}, \mathbb{Z}/N\mathbb{Z}) = H^i_{\text{qproét}}(\widetilde{C}, \mathbb{Z}/N\mathbb{Z}) = \varinjlim_{C' \to C} H^i(C', \mathbb{Z}/N\mathbb{Z})$$

by [32, Proposition 14.9]. Since we have $H^i(C', \mathbb{Z}/N\mathbb{Z}) = 0$ for $i > 2$ by [9, Proposition 8.4.1], this proves the statement for $i > 2$. The same Proposition says that

$$H^2_{\text{ét}}(C', \mathbb{Z}/N\mathbb{Z}) = (\text{Pic}(C')/N)(-1) = \mathbb{Z}/N\mathbb{Z}(-1)$$

via the Kummer sequence and the degree map. It thus suffices to observe that for every line bundle L on C' with Albanese A', the pullback along $[N] : A' \to A'$ kills $\deg L \bmod N$.

By [32, Proposition 3.40.1/2], we have

$$H^i_v(\widetilde{C}, \mathcal{O}^+/p^n) = \varinjlim_{C' \to C} H^i_{\text{ét}}(C', \mathcal{O}^+/p^n).$$

By the Primitive Comparison Theorem [34, Theorem 5.1], we have

$$H^i_{\text{ét}}(C' \times Y) = H^i_{\text{ét}}(C', \mathbb{Z}/p^n) \otimes \mathcal{O}_K/p^k.$$

Since $\varinjlim_{C' \to C} H^i_{\text{ét}}(C', \mathbb{Z}/p^n) = 0$, we deduce that

$$H^i_v(\widetilde{C}, \mathcal{O}^+/p^n) \stackrel{a}{=} 0.$$

The desired statement follows in the limit using that the v-site is replete. □

Corollary 3.2. $\text{Pic}(\widetilde{C})$ *is a divisible group.*

Proof. Immediate from the Kummer sequence. □

3.2. **Line bundles via $\overline{\mathcal{O}}^\times$.**

Lemma 3.3. *We have a short exact sequence*
$$0 \to A^{\mathrm{tt}}(K) \to A(K) \to H^1_{\text{ét}}(C, \overline{\mathcal{O}}^\times) \to \mathbb{Q} \to 0$$

Proof. The long exact sequence
$$H^1_{\text{ét}}(C, \mathcal{O}^{\times,\mathrm{tt}}) \to H^1_{\text{ét}}(C, \mathcal{O}^\times) \to H^1_{\text{ét}}(C, \overline{\mathcal{O}}^\times)$$
can be identified with
$$A^{\mathrm{tt}} \to \mathrm{Pic}(C) \to H^1_{\text{ét}}(C, \overline{\mathcal{O}}^\times)$$
by [14, Theorem 4.2.1]. Here we recall that in the case of curves, the Picard group sits in an exact sequence
$$0 \to A(K) \to \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

We claim that the first of these two sequences becomes exact after tensoring with $\mathbb{Q}$: For this we consider the exponential sequence
$$0 \to \mathcal{O} \to \mathcal{O}^\times \otimes \mathbb{Q} \to \mathcal{O}^\times/\mathcal{O}^{\times,\mathrm{tt}} \to 0,$$
which reduces us to consider the boundary map in
$$H^1_{\text{ét}}(C, \mathcal{O}^\times) \otimes \mathbb{Q} \to H^1_{\text{ét}}(C, \overline{\mathcal{O}}^\times) \to H^2_{\text{ét}}(C, \mathcal{O}).$$
The last term vanishes since $H^2_{\text{ét}}(C, \mathcal{O}) = H^2_{\mathrm{an}}(C, \mathcal{O}) = 0$ as C is a curve. We now consider the morphism from the first sequence tensored with $\mathbb{Q}$ to the second morphism tensored with $\mathbb{Q}$ and use that $A(K)/A^{\mathrm{tt}}(K)$ and $H^1_{\text{ét}}(C, \overline{\mathcal{O}}^\times)$ are both uniquely divisible. ☐

3.3. **Picard group of universal covers of curves.** We can now obtain our desired description of $\mathrm{Pic}(\widetilde{C})$: This answers Question 1.1 in this context:

Proposition 3.4. *We have a short exact sequence*
$$0 \to \varinjlim_{C' \to C} A'(K)^{\mathrm{tt}} \to \varinjlim_{C' \to C} \mathrm{Pic}(C') \to \mathrm{Pic}(\widetilde{C}) \to 0$$
where A' is the Jacobian of C'.

Proof. By Corollary 3.2, we have
$$\mathrm{Pic}(\widetilde{C}) = \mathrm{Pic}(\widetilde{C}) \otimes \mathbb{Q}.$$
Therefore the exponential exact sequence is of the form
$$H^1(\widetilde{C}, \mathcal{O}) \to H^1(\widetilde{C}, \mathcal{O}^\times) \to H^1(\widetilde{C}, \overline{\mathcal{O}}^\times) \to H^2(\widetilde{C}, \mathcal{O}).$$
The outer terms vanish by Proposition 3.1. We are thus reduced to considering $\overline{\mathcal{O}}^\times$.

For this we have by [14, Proposition 3.40.2]
$$H^1_{\text{ét}}(\widetilde{C}, \overline{\mathcal{O}}^\times) = \varinjlim_{C' \to C} H^1_{\text{ét}}(C', \overline{\mathcal{O}}^\times).$$
The desired result now follows from Lemma 3.3 in the colimit over all $C' \to C$. ☐

Remark 3.5. This should also work in characteristic p, by reducing to the abeloid case.

Corollary 3.6. *If $K = \mathbb{C}_p$, then we have $\mathrm{Pic}(\widetilde{C}) = \mathbb{Q}$ via the degree map.*

Proof. This follows from Proposition 3.4. ☐

By Theorem 2.4, the Proposition implies the following stronger statement:

Theorem 3.7. *Let C be a connected smooth proper curve over K. Then the degree map induces a short exact sequence*
$$0 \to \varinjlim_{C' \to C} J_{C'}/J_{C'}^{\mathrm{tt}} \to \mathbf{Pic}_{\widetilde{C}} \xrightarrow{\deg} \underline{\mathbb{Q}} \to 0.$$
where $J_{C'}$ is the Jacobian of C'.

Remark 3.8. By [16, Corollary 4.7, proof of Lemma 5.26], for any two abelian varieties A and A' we have

$$\mathrm{Hom}(A/A^{\mathrm{tt}}, A'/A'^{\mathrm{tt}}) = \mathrm{Hom}(\widetilde{A}, \widetilde{A}).$$

In particular, there is an isomorphism if and only if A and A' have finite étale covers which are p-adically close in the Siegel moduli space. We interpret this as saying that the factor of A'/A'^{tt} remembers "the isogeny class of the Jacobian of C' modulo p".

While this is a bit vague in general, we can be more precise if C' has good reduction: Let $\overline{A}$ be the Jacobian of the special fibre. Then as explained in [15, §5], we have

$$J(C')/J(C')^{\mathrm{tt}} = \overline{A}^{\diamondsuit} \otimes \mathbb{Q}$$

where $\overline{A}^{\diamondsuit} \otimes \mathbb{Q}$ is a certain v-sheaf which remembers the isogeny class of the abelian variety $\overline{A}$ in a fully faithful way.

As pointed out by Litt [24], a result of Bogomolov–Tschinkel [4, Theorem 1.7] implies that if $p \geq 5$ and C is a hyperelliptic curve of good reduction, then for every abeloid variety of good reduction B over $\overline{\mathbb{F}}_p$, the sheaf $B^{\diamondsuit} \otimes \mathbb{Q}$ appears in the $\mathbf{Pic}_{\widetilde{C}}$. The same argument for B^n for $n \to \infty$ shows that $B^{\diamondsuit} \otimes \mathbb{Q}$ appears with infinite multiplicity. Since for any abeloid variety A over $\mathbb{C}_p$, the space A/A^{tt} is determined by the good reduction part and the torus part of the semi-stable reduction [16, Theorem 1.1], shows:

Corollary 3.9. *Let $p \geq 5$, let $K = \mathbb{C}_p$ and let C be any curve isogeneous to a hyperelliptic curve of good reduction. Then*

$$\mathbf{Pic}^0_{\widetilde{C}} = \Big(\overline{\mathcal{O}}^{\times,\mathrm{tt}}/p^{\mathbb{Q}} \times \bigoplus_{B \in \mathcal{A}(\overline{\mathbb{F}}_p)} B^{\diamondsuit} \otimes \mathbb{Q}\Big)^{\oplus \mathbb{N}}$$

where $\mathcal{A}(\overline{\mathbb{F}}_p)$ is the set of isomorphism classes of abelian varieties over $\overline{\mathbb{F}}_p$ of any dimension.

Here the first factor accounts for all totally degenerate abeloids A, for which A/A^{tt} is isomorphic to a finite sum of copies of E/E^{tt} for the Tate curve $E = \mathbb{G}_m/p^{\mathbb{Z}}$.

Conjecture 3.10. $\mathbf{Pic}^0_{\widetilde{C}}$ is of this form for any curve of genus ≥ 2 over $\mathbb{C}_p$.

Remark 3.11. In [16], we asked whether $\widetilde{C}$ was locally constant in the moduli space of curves $\mathcal{M}_g(K)$ of genus g. For $g \geq 2$, we even optimistically asked if *all* $\widetilde{C}$ for C over K might be isomorphic, which would produce a strong uniformisation result. In order to get a feeling for whether this has a chance of being true, a natural approach is to compare invariants of $\widetilde{C}$ for different C. In this light, we note that over $\mathbb{C}_p$ Theorem 3.7 seems consistent with both conjectures when Conjecture 3.10, thus producing some weak evidence that they might hold.

On the other hand, for fields other than $\mathbb{C}_p$, Theorem 3.7 can be used to prove the following negative result:

Corollary 3.12. *Let $K \supsetneq \mathbb{C}_p$ be an algebraically closed field. Let C_1 be any hyperbolic curve of good reduction over K for which the reduction of the Jacobian over k is not already defined over $\overline{\mathbb{F}}_p$. Let C_2 be any curve over K that is already defined over $\mathbb{C}_p$. Then*

$$\widetilde{C}_1 \neq \widetilde{C}_2.$$

Proof. If there was an isomorphism $\widetilde{C}_1 = \widetilde{C}_2$, this would under Theorem 3.7, by passing to connected components of the identity, induce an isomorphism

$$\varinjlim_{C_2' \to C_2} J(C_2')/J(C_2')^{\mathrm{tt}} \xrightarrow{\sim} \varinjlim_{C_1' \to C_1} J(C_1')/J(C_1')^{\mathrm{tt}}.$$

Let B be the Jacobian of the reduction $\overline{C}_1$ over k, then the right hand side has a direct factor given by $B^{\diamondsuit} \otimes \mathbb{Q}$. Consider the projection to this factor, then the above isomorphism induces a surjection

$$\varinjlim_{C_2' \to C_2} J(C_2')/J(C_2')^{\mathrm{tt}} \to B^{\diamondsuit} \otimes \mathbb{Q}.$$

Let now $C_{2,\mathbb{C}_p}$ be the model of C_2 over $\mathbb{C}_p$, then $C_{2,\mathbb{C}_p,\text{fét}} = C_{2,\text{fét}}$. Consequently, this map is the direct limit of morphisms

$$\phi : A'/A'^{\text{tt}} \to B^{\diamond} \otimes \mathbb{Q}$$

where A' is an abelian variety that is already defined over $\mathbb{C}_p$. Let B' be the abelian part of the semi-stable reduction of A' over k, then by [16, Theorem 5.24] any such morphism induces a morphism $B' \to B$ over k that determines ϕ uniquely. But since B' is defined over $\bar{\mathbb{F}}_p$, this will always factor over the largest isogeny factor of B that is already defined over $\bar{\mathbb{F}}_p$. By assumption on C_1, this is a proper subvariety of B. Consequently, again by fully faithfulness of the functor $B \mapsto B^{\diamond} \otimes \mathbb{Q}$ on the isogeny category, the same is true for the image the direct limit of ϕ. This is a contradiction to this being a surjection. □

3.4. **Characteristic p.** We now prove Theorem 3.7 in the case that K has characteristic p:

Theorem 3.13. *Let C be a connected smooth proper curve over K. Then the degree map induces a short exact sequence*

$$0 \to \varinjlim_{C' \to C} J_{C'}/J_{C'}^{\text{tt}} \to \mathbf{Pic}_{\widetilde{C}} \xrightarrow{\deg} \underline{\mathbb{Q}} \to 0.$$

where $J_{C'}$ is the Jacobian of C' considered as a v-sheaf.

This is achieved by the following series of propositions: Let C be a connected smooth projective curve over an algebraically closed non-archimedean field of characteristic p.

Proposition 3.14. *Let $\widetilde{\pi} : \widetilde{C} \to \mathrm{Spa}(K)$ be the projection, then*

$$\mathbf{Pic}_{\widetilde{C}} = R^1\widetilde{\pi}_*\mathcal{O}^\times = R^1\widetilde{\pi}_*(\mathcal{O}^\times/\mathcal{O}^{\times,\text{tt}}).$$

[I think we don't even really need this]

Proof. Since $\widetilde{C}$ is perfectoid, we can by [15, Theorem 2.18] compute both sheaves in the pro-étale topology. Here we can use the sequence

$$\mathbb{Q}_p \to 1 + \mathfrak{m} \to \mathcal{O}^\sharp,$$

which reduces us to showing $R^i\widetilde{\pi}_*\mathbb{Q}_p = 0$ and $R^i\widetilde{\pi}_*\mathbb{Z}/N\mathbb{Z} = 0$ for $N \in \mathbb{N}$ and $R^i\widetilde{\pi}_*\mathcal{O}^\sharp = 0$ for $i = 1, 2$. These can be seen by arguing as in [14, Proposition 4.6], using the Artin–Schreier sequence and the Primitive Comparison Theorem in characteristic p. □

Proposition 3.15. *The natural map $\varinjlim_{C' \to C}(\mathbf{Pic}_{C'})^{\diamond} \to \mathbf{Pic}_{\widetilde{C}}$ is surjective.*

Proof. By Proposition 3.14, we have

$$H^1(\widetilde{C} \times T, \mathcal{O}^\times) = H^1(\widetilde{C} \times T, \overline{\mathcal{O}}^\times),$$

at least after sheafification. The right hand side equals

$$\varinjlim_{C' \to C} H^1(C' \times T', \overline{\mathcal{O}}^\times)$$

where T' ranges through the affinoid smooth rigid spaces to which T maps. It therefore suffices to prove that for an affinoid rigid space T, the natural map

$$\mathrm{Pic}(C \times T)[\tfrac{1}{p}] \to H^1(\widetilde{C} \times T, \overline{\mathcal{O}}^\times)$$

is surjective. For this consider the projection map

$$q : C \times T \to C.$$

The Leray sequence applied to the map $\mathcal{O}^\times \to \overline{\mathcal{O}}^\times$ yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(C, q_*\mathcal{O}^\times) & \longrightarrow & H^1(C \times T, \mathcal{O}^\times) & \longrightarrow & H^0(C, R^1q_*\mathcal{O}^\times) & \longrightarrow & H^2(C, q_*\mathcal{O}^\times) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(C, q_*\overline{\mathcal{O}}^\times) & \longrightarrow & H^1(C \times T, \overline{\mathcal{O}}^\times) & \longrightarrow & H^0(C, R^1q_*\overline{\mathcal{O}}^\times) & &
\end{array}$$

The term on the top right vanishes since analytic cohomology of C is concentrated in degrees $[0,1]$. It thus suffices to prove that the outer vertical maps are surjective. For this consider the long exact sequence

$$0 \to q_*(1+\mathfrak{m}) \to q_*\mathcal{O}^\times \to q_*\overline{\mathcal{O}}^\times \to R^1q_*(1+\mathfrak{m}) \to R^1q_*\mathcal{O}^\times \to R^1q_*\overline{\mathcal{O}}^\times.$$

By [15, Lemma 2.14.2], the boundary map is trivial, and by [15, Lemma 2.14.3] the last map is an isomorphism after inverting p. It follows that the morphism on the right is an isomorphism. The map on the left sits in an exact sequence the next term of which is $H^2(C, q_*(1+\mathfrak{m})) = 0$. □

Proposition 3.16. *Let C be a smooth proper curve over an algebraically closed non-archimedean field (K, K^+). Let A be the Jacobian of C considered as a curve over K. Let L be a line bundle on C that is in the image of*

$$H^1_{\text{ét}}(C, 1+\mathfrak{m}\mathcal{O}^+) \to \mathrm{Pic}_{\text{ét}}(C).$$

Then the corresponding point in $A(K,K^+)$ is contained in $\widehat{A}(K,K^+)$.

Proof. We first reduce to the case of $K^+ = \mathcal{O}_K := K^{\circ\circ}$ by by pullback along the map $\mathrm{Spa}(K,\mathcal{O}_K) \to \mathrm{Spa}(K,K^+)$: Let

$$C^\circ := C \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}(K,\mathcal{O}_K) \to C$$

be the base-change. For this we first note that the induces map

$$\mathrm{Pic}(C) \to \mathrm{Pic}(C^\circ)$$

is a bijection: This is because on any cover of C by affinoid opens U, line bundles correspond to locally free $\mathcal{O}(U)$-modules of rank 1, which is independent of $\mathcal{O}^+$. Similarly, the gluing data are in terms of sections of $\mathcal{O}^\times$, which does not see $\mathcal{O}^+$ either. Finally, we note that due to the perfectoidness assumption, $\mathfrak{m}K^+ = \mathfrak{m}\mathcal{O}_K$, so the image of $H^1(C, 1+\mathfrak{m}\mathcal{O}^+)$ in $\mathrm{Pic}(C)$ also agrees with that of $H^1(C^\circ, 1+\mathfrak{m}\mathcal{O}^+)$ in $\mathrm{Pic}(C^\circ)$.

On the other hand, since both A and $\widehat{A}$ are partially proper, we have

$$A(K,K^+) = A(K,\mathcal{O}_K), \quad \widehat{A}(K,K^+) = \widehat{A}(K,\mathcal{O}_K).$$

We can thus restrict to the classical rigid case over $(K,K^+) = (K,\mathcal{O}_K)$. In this case, the result follows from the rigid theory of semi-stable reduction of curves as developed by Bosch–Lütkebohmert [5, §4]:

Namely, let $A := \mathrm{Jac}(C)$ be the Jacobian of C. Then by [5, Theorem 5.1] the semi-stable reduction $\pi : C \to C_k$ of C gives rise to a formal semi-abelian open subgroup $\overline{A}$ over $\mathcal{O}_K$ which reduces to the generalised Jacobian of C_k. Moreover, according to [5, Proposition 5.16, Remark 5.17], the map

$$H^1_{\mathrm{Zar}}(C_k, \pi_*\mathcal{O}^{+,\times}) \to H^1_{\mathrm{an}}(X, \mathcal{O}^\times) = A(K)$$

factors through $\overline{A}(K)$, and the following diagram commutes:

$$\begin{array}{ccc} H^1_{\mathrm{Zar}}(C_k, \pi_*\mathcal{O}^{+,\times}) & \longrightarrow & \overline{A}(K) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{Zar}}(C_k, \mathcal{O}^\times) & \longrightarrow & \mathrm{Pic}(C_k). \end{array}$$

Since $H^1_{\text{ét}}(U, 1+\mathfrak{m}\mathcal{O}^+)$ is p-torsion on affinoids U by [15, Lemma 2.14], we conclude that some p-th power of L comes from $H^1_{\mathrm{Zar}}(C_k, \pi_*(1+\mathfrak{m}\mathcal{O}^+))$, which by the above diagram maps into the formal fibre $\overline{A}_1$ of $\overline{A}$ over $1 \in \mathrm{Pic}(C_k)$. It follows that some p-th power of L lies in $\overline{A}_1(K) \subseteq A(K)$, see also [26, Lemma 5.3.2].

By [14, Proposition 2.19], we have $\overline{A}_1 \subseteq \widehat{A}$. This shows that some p-th power of L lies in $\widehat{A}(K)$, and thus L lies in $\widehat{A}(K)$. □

Proof. Consider the kernel W of the morphism of abelian v-sheaves

$$\mathbf{Pic}_C \to \mathbf{Pic}_{\widetilde{C}}.$$

By Proposition 3.15, we are left to see that W equals A^{tt} where A is the Jacobian.

Using our fixed base-point $x \in C(K)$, we can embed $C \hookrightarrow A$ into the Albanese variety, which we may identify with the Jacobian. By the universal property of the universal cover, we obtain a morphism

$$\widetilde{C} \to \widetilde{A}.$$

Let $\mathrm{Pic}^{\diamondsuit}_A$ be the perfection of the rigid group representing the Picard functor of A, then we get a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \longrightarrow & \mathbf{Pic}_C & \xrightarrow{h} & \mathrm{Pic}_{\widetilde{C}} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A^{\mathrm{tt}} & \longrightarrow & \mathrm{Pic}^{\diamondsuit}_A & \longrightarrow & \mathrm{Pic}_{\widetilde{A}}. \end{array}$$

It follows that $A^{\mathrm{tt}} \subseteq W$.

To get the other inclusion, it suffices to prove that for any algebraically closed extension $T = (L, L^+)$ over (K, K^+), an (L, L^+)-point $x \in \mathbf{Pic}_C(L, L^+)$ is in the kernel of h if and only if it factors through $A^{\mathrm{tt}}(L, L^+)$.

To see this, let $C_L := C \times \mathrm{Spa}(L, L^+)$. Since the transition maps

$$H^1(C_L, \mathcal{O}^\times/\mathcal{O}^{\times,\mathrm{tt}}) \to H^1(C'_L, \mathcal{O}^\times/\mathcal{O}^{\times,\mathrm{tt}})$$

are injective, it follows that x is in the image of

$$H^1(C_L, \mathcal{O}^{\times,\mathrm{tt}}) \to H^1(C_L, \mathcal{O}^\times).$$

Thus some N-th power of x is in the image of

$$H^1(C_L, 1 + \mathfrak{m}) \to H^1(C_L, \mathcal{O}^\times).$$

But this factors through $\widehat{A}(L, L^+) \subseteq A(L, L^+)$ by Proposition 3.16. Since $A^{\mathrm{tt}}(L, L^+)$ are precisely those elements for which some power is in $\widehat{A}(L, L^+)$, it follows that $x \in A^{\mathrm{tt}}(L, L^+)$, as we wanted to see. □

3.5. **Applications.**

3.5.1. *Counter-examples to uniformisation over* $\mathbb{C}_p^\flat$. As our main application within this article, we deduce that the analogue of the Uniformisation Conjecture over $\mathbb{C}_p^\flat$ is false:

Corollary 3.17. *Let C_1 be any curve over $\mathbb{C}_p^\flat$ whose Jacobian does not have good reduction, e.g. a Mumford curve. Let C_2 be any curve over K that is already defined over $\overline{\mathbb{F}}_p \subseteq \mathbb{C}_p^\flat$. Then*

$$\widetilde{C}_1 \neq \widetilde{C}_2.$$

Proof. Let $C_{2,\overline{\mathbb{F}}_p}$ be the model of C_2 over $\overline{\mathbb{F}}_p$. Then by permanence of π_1 under algebraically closed base-change, we have

$$\pi_1(C_{2,\overline{\mathbb{F}}_p}) = \pi_1(C_2),$$

i.e. any connected finite étale cover is also already defined over $\overline{\mathbb{F}}_p$. It follows from this that every finite étale cover of C_2 has good reduction. In particular, we have

$$\mathbf{Pic}_{\widetilde{C}_2} = \varinjlim_{C' \to C_{2,\overline{\mathbb{F}}_p}} J(C')^{\diamondsuit}$$

where $J(C')$ is the Jacobian considered as a scheme over k.

On the other hand, by assumption the Jacobian A of C_1 has bad reduction. It follows that $\mathbf{Pic}_{\widetilde{C}_2}$ has a direct factor of the form $A/\widehat{A}$. Consequently, any isomorphism $\widetilde{C}_1 \xrightarrow{\sim} \widetilde{C}_2$ would induce a surjective morphism

$$\varinjlim_{C' \to C_{2,\overline{\mathbb{F}}_p}} J(C')^{\diamondsuit} \to A/\widehat{A}.$$

But since the Jacobians on the left have good reduction, any such morphism is trivial by the argument in the proof of [16, Lemma 5.26]. □

3.5.2. *The degree of v-vector bundles.* As a second application, in characteristic 0 we can use the description of the Picard group at infinite level to define a natural notion degree of v-vector bundles. This has applications in the context of the p-adic Simpson correspondence: Namely, in order to consider the open question which Higgs bundles correspond to representations, one would as an intermediate step like to define v-vector bundles of degree 0. In the case of curves this is an analogue of the complex analytic condition to have "vanishing Chern classes").

Definition 3.18. Let C be a smooth proper curve over K.

(1) Let L be a v-line bundle on C. Then its pullback $\widetilde{L}$ to $\widetilde{C}$ is analytic because $\widetilde{C}$ is perfectoid. We define the degree of L to be the image of L under the degree map on $\widetilde{C}$ in Theorem 3.7. More generally, this defines a notion of the (locally constant) degree of a v-line bundles on $X \times Y$ where Y is any diamond over K.

(2) Let V be a v-vector bundle on C. Its degree is defines as the degree of $\det V$, which is a v-line bundle on C.

(3) A v-vector bundle is called (semi-)stable if for every sub-v-vector bundle $W \subseteq V$, we have $\deg W < \deg V$ (respectively, $\deg W \leq \deg V$).

Remark 3.19. One could give a more at hoc definition of the degree of a line bundle by choosing a Hodge–Tate splitting and an exponential, which induces a splitting of the sequence

$$0 \to \mathrm{Pic}_{\text{ét}}(C) \to \mathrm{Pic}_v(C) \to H^0(C, \widetilde{\Omega}^1) \to 0$$

and then define the degree of a v-line bundle to be the degree of its image in $\mathrm{Pic}_{\text{ét}}(C)$. However, this definition is artificial and has several obvious disadvantages: One has to check independence of choice, and properties like functoriality and compatibility in families are harder to check. However, it is easy to see that the definition agrees with the one above.

The following is easy to deduce from the given definition that this satifies the usual compatibilities of the degree. For example:

Lemma 3.20. *Let $f : C' \to C$ be a finite flat morphism of curves of degree n. Then for any v-line bundle V on C, we have* $\deg(f^*V) = n \deg(V)$.

Proof. Via pullback, it is clear from the definition that it suffices to check this for the morphism of universal covers $\widetilde{C}' \to \widetilde{C}$. Here any line bundle on $\widetilde{C}$ descends to an analytic vector bundle on a finite étale cover of C, so the result follows from the algebraic case. □

Remark 3.21. One might ask whether a version of the Riemann–Roch Theorem holds in this setting. This is arguably the case, but the degree doesn't enter in this: In general, it only makes sense to ask for the the dimension of v-cohomology, for which one always have constant Euler characteristic. Namely, for any v-line bundle, one can show that

$$\dim_K H^0_v(X, L) - \dim_K H^1_v(X, L) + \dim_K H^2_v(X, L) = 2(1 - g)$$

regardless of degree.

4. Picard groups of perfectoid tori

We now move on to discussing Picard groups in the non-compact case of tori. While we are interested in this question in and of itself, it will also serve as an input into our computation of $\mathrm{Pic}(\widetilde{A})$ in the next sections.

For the complex manifold $\mathbb{G}_{m,\mathbb{C}} := \mathbb{P}^1_{\mathbb{C}}\backslash\{0,\infty\}$, and any $r > 0$, we have

$$\mathrm{Pic}_{\mathrm{hol}}(\mathbb{G}^r_{m,\mathbb{C}}) = \mathrm{Pic}_{\mathrm{top}}(\mathbb{G}^r_{m,\mathbb{C}}) = 1.$$

The analogous statement also holds in rigid geometry for the rigid torus $\mathbb{G}^r_m$:

Theorem 4.1 ([9, Thm 6.3.3])**.** $\mathrm{Pic}_{\mathrm{an}}(\mathbb{G}^r_m) = \mathrm{Pic}_{\text{ét}}(\mathbb{G}^r_m) = 1$.

The goal of this subsection is to show that the analogous statement for the perfectoid torus does not hold. Indeed, as a qualitative statement this, this can be deduced from [17, Theorem 1.3.2] which says that we have more line bundles for the finer v-topology:

$$\mathrm{Pic}_v(\mathbb{G}_m) = H^0(\mathbb{G}_m, \Omega^1)(-1).$$

While for proper rigid spaces, these additional v-topological line bundles can be explained in terms of descent data on the trivial line bundle on the universal cover, the situation for the Stein space $\mathbb{G}_m$ is different: Namely, the Cartan–Leray sequence for the universal p-adic cover $\widetilde{\mathbb{G}}_m \to \mathbb{G}_m$ yields an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_p(1), \mathcal{O}^\times(\widetilde{\mathbb{G}}_m)) \to \mathrm{Pic}_v(\mathbb{G}_m) \to \mathrm{Pic}_v(\widetilde{\mathbb{G}}_m)^{\mathbb{Z}_p}$$

the first term of which equals $\mathrm{Hom}(\mathbb{Z}_p(1), 1+\mathfrak{m}) = K(-1)$ via the logarithm map, and can be identified with the subspace given by the invariant differentials $K\frac{dT}{T} \subseteq H^0(\mathbb{G}_m, \Omega^1)$ via [17, Corollary 4.4]. Since these only account for a small part of $H^0(\mathbb{G}_m, \Omega^1) = \mathcal{O}(\mathbb{G}_m)\frac{dT}{T}$, we see that there have to be non-trivial line bundles in $\mathrm{Pic}_v(\widetilde{\mathbb{G}}_m)$ that descend to $\mathbb{G}_m$. The first goal of this section is to describe these new v-line bundles quantitatively.

The second goal is to show that these new line-bundles disappear when passing from line bundles to torsors under

$$\mathcal{O}^\times[\tfrac{1}{p}] := \varinjlim_{[p]} \mathcal{O}^\times.$$

Indeed, the group $H^1(\widetilde{T}, \mathcal{O}^\times[\frac{1}{p}])$ will turn out to be trivial.

4.1. **Invertible functions.** Throughout this section, let K be an algebraically closed perfectoid field of arbitrary characteristic, and let T be a torus of rank r over K, so that $T \cong \mathbb{G}^r_m$. Let $N = \mathrm{Hom}(T, \mathbb{G}_m)$ and $N^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$ be the character and cocharacter lattice, respectively. We begin in this subsection with some easy preparations.

Definition 4.2. Choose a basis $X_1, \dots, X_r$ of N. For any $n \in N$ with $n = X_1^{m_1} \cdots X_r^{m_r}$, we write $|n| = m_1, \dots, m_r$. By the standard cover of T we mean the cover over $s \in \mathbb{N}$ of

$$T_s := T(|\pi|^s \leq |X_i| \leq |\pi|^{-s} \text{ for } i = 1, \dots, r),$$

where we recall that $\pi \in \mathfrak{m}$ is a pseudo-uniformiser. Let $\widetilde{T}_s$ be the pullback of T_s to $\widetilde{T}$.

Lemma 4.3. *Let S be any diamond. Let $R = \mathcal{O}(S)$ and $R^+ = \mathcal{O}^+(S)$. Let $\underline{N}$ be the constant sheaf where $N = \mathrm{Hom}(T, \mathbb{G}_m)$ is endowed with the discrete topology.*

(1) $\mathcal{O}^+(T \times S) = R^+$,
(2) $\mathcal{O}^+(\widetilde{T} \times S) = R^+$,
(3) $\mathcal{O}^\times(T \times S) = \underline{N}(S) \times R^\times$,
(4) $\mathcal{O}^\times(\widetilde{T} \times S) = \underline{N}[\frac{1}{p}](S) \times R^\times$,
(5) $\overline{\mathcal{O}}^\times(T \times S) = \underline{N}[\frac{1}{p}](S) \times \overline{\mathcal{O}}^\times(S)$,
(6) $\overline{\mathcal{O}}^\times(\widetilde{T} \times S) = \underline{N}[\frac{1}{p}](S) \times \overline{\mathcal{O}}^\times(S)$.

Proof. (1) The statement is local on S, so we can reduce to the case that S is totally disconnected, in particular affinoid. By induction, we may reduce to $T = \mathbb{G}_m$. Then f can be written as a power series $f = \sum_{n\in\mathbb{Z}} a_n X^n$ for some $a_n \in R$. We claim that unless $n = 0$, we have $a_n = 0$. We can check this at every point of S, which reduces us to the case that $(R, R^+) = (C, C^+)$ is a non-archimedean field.

The valuation ring C^+ is then microbial, so there is a unique rank 1 generization $(C, \mathcal{O}_C)$. Since $\mathrm{Spa}(C, \mathcal{O}_C) \subseteq \mathrm{Spa}(C, C^+)$, it suffices to prove the statement in this case. This puts us in a rigid geometric situation, where one can argue as usual:

For $|x| \gg 0$, we have $|\sum_{n\in\mathbb{Z}} a_n x^n| = |\sum_{n=0}^{\infty} a_n x^n|$. Thus on annuli of inner radius $\gg 0$ and outer radius $r \gg 0$, the function f attains its supremum given by the Gauss norm $\sup_{n\geq 0} |a_n| r^n$. Unless $a_n = 0$ for all $n > 0$, this diverges. The same argument for $|x^{-1}| \gg 0$ shows that $a_n = 0$ for all $n < 0$. Thus the only bounded functions are the constant ones.
(2) The statement is local on S, so we can reduce to the case that S is affinoid perfectoid with $S = \mathrm{Spa}(S, S^+)$. By induction, we may reduce to $T = \mathbb{G}_m$. Then f can be written as a perfectoid power series with coefficients in R:

$$f = \sum_{n\in\mathbb{Z}[\frac{1}{p}]} a_m X^m.$$

We claim that $a_n = 0$ unless $n = 0$. To see this, we first conclude from [33, Lemma 6.4] that

$$\mathcal{O}^+(\widetilde{T}_s \times S) \overset{a}{=} R^+\langle X^{1/p^\infty}, Y^{1/p^\infty}\rangle/(X^{1/p^\infty}Y^{1/p^\infty} - p^{s/p^\infty}).$$

This shows that for f to be in $\mathcal{O}^+(\widetilde{T} \times S)$, we need to have $\mathfrak{m} \cdot a_n \subseteq p^{s\cdot|n|}\mathcal{O}_K$ for all $s \in \mathbb{N}$, which implies $a_n = 0$ unless $n = 0$.
(3) We follow [9, Lemma 6.3.1] (which is the case of $S = \mathrm{Spa}(K, \mathcal{O}_K)$): Namely, we instead prove that for any $s \geq 0$,

$$\mathcal{O}^\times(T_s \times S) = \underline{N}(S) \times R^\times \cdot (1 + \mathfrak{m}\mathcal{O}^+(T_s \times S))). \tag{3}$$

The desired statement then follows in the limit $s \to \infty$ using the first part.

We can again reduce to the case that S is totally disconnected and that $T = \mathbb{G}_m$. Let

$$\varphi = \sum_{n=0}^{\infty} a_n \pi^{-sn} X^n + \sum_{n=1}^{\infty} a_{-n}\pi^{-sn} X^{-n} \in \mathcal{O}^\times(T_s \times S),$$

where $a_n \in R$ with $\|a_n\| \to 0$ for $n \to \infty$ and $\|a_n\| \to 0$ for $n \to -\infty$. For every $n \in \mathbb{N}$, we define

$$S_n = S(0 \neq |a_n| > |a_m| \text{ for all } n \neq m \in \mathbb{Z}).$$

We claim that the S_n form a disjoint open cover of S. This would prove the claim, since one each S_n, the element φ is then be of the form $a_n \cdot X^n \cdot (\sum (a_n^{-1} a_m) X^{m-n})$.

It is clear that S_n and S_m are disjoint whenever $n \neq m$. To see that S_n is open, we first introduce the auxiliary intermediate subspace

$$S_n \subseteq S'_n := S(0 \neq |a_n| \geq |a_m| \text{ for all } m \in \mathbb{Z}).$$

It is clear that we have

$$S = \bigcup_{n\in\mathbb{Z}} S'_n.$$

We claim that S'_n is affinoid open: For this we first note that φ being a unit, there are $b_n \in R$ such that $\sum_{n=-\infty}^{\infty} a_n b_n = 1$. Since there are only finitely many n with $\|a_n b_n\| \geq 1$, this implies that there is a finite index set $I \subseteq \mathbb{Z}$ such that $1 \in (a_i | i \in I)$. Thus

$$S''_n := S(0 \neq |a_n| \geq |a_i| \text{ for all } i \in I) \subseteq S$$

is affinoid open. In particular, it is compact, so that $|a_n|$ can be bounded from below. This shows that $|a_n| \geq \|a_m\| \geq |a_m|$ for all but finitely many $m \in \mathbb{Z}$. Adding these to I, we see that

$$S'_n = S''_n(0 \neq |a_n| \geq |a_i| \text{ for all } i \in I)$$

is indeed affinoid open.

Finally, we claim that in fact,

$$S'_n = S_n.$$

Indeed, suppose $x \in S_n$. Then x is in the image of a morphism

$$x : \mathrm{Spa}(C, C^+) \to S'_n$$

for some non-archimedean field extension (C, C^+) of $(K, \mathcal{O}_K)$. Let x' be the rank-1-generization, corresponding to a pair $(C, \mathcal{O}_C)$. it suffices to prove that $x' \in S_n$, since x' being a generization, $|a_n(x')| > |a_m(x')|$ implies $|a_n(x)| > |a_m(x)|$.

We are now in a position to apply [9, Lemma 6.3.1], which guarantees that the pullback of φ to $T'_s = T_s \times \mathrm{Spa}(C, \mathcal{O}_C)$ is in $X^k \cdot C^\times \cdot (1 + \mathfrak{m}_C \mathcal{O}^+(T'_s))$ for some $k \in \mathbb{Z}$. The condition that $x \in S'_n$ now implies that $k = n$. For any $m \neq n$, this description now shows that $a_m = a_n \cdot z$ for some $z \in \mathfrak{m}_L$, which implies $|a_n(x)| > |a_m(x)|$. Thus $x \in S_n$, as desired.

This finishes the proof that $S = \cup S_n$ is a disjoint open cover.

(4) With reduction steps like for 3, we reduce to showing

$$\mathcal{O}^\times(\widetilde{T}_s \times S) = \underline{N}[\tfrac{1}{p}](S) \times R^\times \cdot (1 + \mathfrak{m}\mathcal{O}^+(\widetilde{T}_s \times S))$$

for S totally disconnected. Let $T_s^{(n)} = \mathrm{Spa}(K\langle \pi^{s/p^n} X^{1/p^n}, \pi^{s/p^n}/X^{1/p^n}\rangle)$. This is isomorphic to T_{s/p^n}, but with parameter X^{1/p^n} rather than X. Then the natural map

$$\varinjlim_n \mathcal{O}(T_s^n \times S) = \varinjlim_n R\langle \pi^{s/p^n} X^{1/p^n}, \pi^{s/p^n}/X^{1/p^n}\rangle \to \mathcal{O}(\widetilde{T}_s \times S)$$

has dense image. In particular, any unit $\varphi \in \mathcal{O}(\widetilde{T}_s \times S)^\times$ is of the form $\varphi_k + h$ for some $\varphi_k \in \mathcal{O}(T_s^{(k)} \times S)$ and $h \in \mathcal{O}(\widetilde{T}_s \times S)^{\circ\circ}$, which implies $\varphi_k \in \mathcal{O}(T_s^{(k)} \times S)^\times$. By the third part, locally on S, we have

$$\varphi_k \in \mathcal{O}(\widetilde{T}_s^{(k)} \times S)^\times = R^\times \times \tfrac{1}{p^k} N(S) \times (1 + \mathcal{O}(T_s^{(k)} \times S)^{\circ\circ}).$$

When we now take the colimit $k \to \infty$ and choose better approximations of φ, then by comparing coefficients we see that the factors in $R^\times$ and $\frac{1}{p^k} N(S)$ stabilise. Thus the φ_k converge to an element of

$$R^\times \times \underline{N}[\tfrac{1}{p}](S) \times (1 + \mathcal{O}(\widetilde{T}_s \times S)^{\circ\circ}),$$

as desired.

(5) Will follow from part 6 by taking $\mathbb{Z}_p(1)$-invariants.

(6) We may reduce to the case that S is affinoid perfectoid. Then so is $\widetilde{T}_s \times S$, and we thus have a short exact exponential sequence

$$0 \to \mathcal{O}(\widetilde{T}_s \times S) \to \mathcal{O}^\times(\widetilde{T}_s \times S)[\tfrac{1}{p}] \to \overline{\mathcal{O}}^\times(\widetilde{T}_s \times S) \to 0.$$

Part 3 now implies that the quotient equals

$$\begin{aligned}\overline{\mathcal{O}}^\times(\widetilde{T}_s \times S) =& \underline{N}[\tfrac{1}{p}](S) \times R^\times[\tfrac{1}{p}] \cdot (1 + \mathfrak{m}\mathcal{O}^+(\widetilde{T}_s \times S))[\tfrac{1}{p}]/\exp(\mathcal{O}(\widetilde{T}_s \times S)) \\ =& \underline{N}[\tfrac{1}{p}](S) \times R^\times[\tfrac{1}{p}]/(1 + \mathfrak{m}R^+)[\tfrac{1}{p}]\end{aligned}$$

which equals $\underline{N}[\frac{1}{p}](S) \times \overline{\mathcal{O}}^\times(S)$ as desired. □

Lemma 4.4. *Let Y be a diamond and let Z be (the diamond associated to) an affinoid adic space. Then*

$$\mathrm{Map}(\widetilde{T} \times Y, Z) = \mathrm{Map}(T \times Y, Z) = \mathrm{Map}(Y, Z).$$

Proof. Let $Z = \mathrm{Spa}(S, S^+)$, then any map $f : T \times Y \to Z$ corresponds to a homomorphism

$$S^+ \to \mathcal{O}^+(T \times Y).$$

But by Lemma 4.3, we have $\mathcal{O}^+(T \times Y) = \mathcal{O}^+(Y)$. □

4.2. **Picard groups of relative tori.** The goal of this subsection is to prove the following result, which is useful in the context of Raynaud extensions:

Proposition 4.5. *Let X be the generic fibre of a smooth affine formal scheme over $\mathcal{O}_K$. Let T be a torus over K. Then pullback along $T \times X \to X$ defines an isomorphism*

$$\mathrm{Pic}(T \times X) = \mathrm{Pic}(X).$$

The proof is essentially just a relative improvement of [9, Lemma 4.7.3 and Thm 6.3.3]. We start by replacing T with increasingly better affinoid approximation:

Lemma 4.6. *Let X be the generic fibre of a smooth affine formal scheme over $\mathcal{O}_K$. Then*

(1) $\mathrm{Pic}_{\mathrm{an}}(X \times \mathrm{Spa}(K\langle X\rangle)) = \mathrm{Pic}_{\mathrm{an}}(X)$,

(2) $\mathrm{Pic}_{\mathrm{an}}(X \times \mathrm{Spa}(K\langle X^{\pm 1}\rangle) = \mathrm{Pic}_{\mathrm{an}}(X)$.

Remark 4.7. The condition that X has good reduction is necessary, see [11, §4.2].

Proof. If K is discretely valued, this is the content of [11, §4.2-3]. Using [26, Lemma 6.2.4] and [15, Lemma 3.6], the proof goes through in general: Let $\overline{X}$ be the special fibre over k, then using the lemma repeatedly, we have

$$\mathrm{Pic}(X \times \mathrm{Spa}(K\langle X\rangle)) = \mathrm{Pic}(\overline{X} \times \mathbb{G}_{a,k}) = \mathrm{Pic}(\overline{X}) = \mathrm{Pic}(X),$$

$$\mathrm{Pic}(X \times \mathrm{Spa}(K\langle X^{\pm 1}\rangle)) = \mathrm{Pic}(\overline{X} \times \mathbb{G}_{m,k}) = \mathrm{Pic}(\overline{X}) \times H^1(\overline{X}, \mathbb{Z}) = \mathrm{Pic}(\overline{X}) = \mathrm{Pic}(X).$$

where in each case, the second equation follows from [13, Prop 2.2.1-3]. □

Lemma 4.8. *Let X be the generic fibre of a smooth affine formal scheme over $\mathcal{O}_K$. Then pullback along $T_s \times X \to X$ defines an isomorphism*

$$\mathrm{Pic}(X) = \mathrm{Pic}(T_s \times X).$$

Proof. Let us change notation and write $T = \mathbb{G}_m$, so that the torus we look at becomes T^r.

We prove the statement by induction on r. The case of $r = 0$ is clear. Let now $r \geq 0$ and assume we know the statement for r. Let L be a line bundle on $T_s^r \times T_s \times X$.

Recall that T_s is a one-dimensional rigid annulus $D(a, b)$ of inner radius a and outer radius b. Let $D(a)$ be the inner boundary and consider the restriction L_0 of L to $T_s^r \times D(a) \times X$. Since $D(a)$ has a smooth affine formal model, so does $D(a) \times X$. The induction hypothesis therefore tells us that L_0 comes from a line bundle on $\mathrm{Pic}(D(a) \times X)$. By Lemma 4.6 we see that

$$\mathrm{Pic}(D(a) \times X) = \mathrm{Pic}(X)$$

via the projection map, thus L_0 is the pullback of a line bundle L_X on X.

Let now $B(a) \subseteq \mathbb{A}^1$ be the closed disc centered at 0 of radius a. Then we can glue L_0 on $T_s^r \times T_s \times X$ along the boundary $T_s^r \times D(a) \times X$ to the line bundle on $T_s \times B(a) \times X$ defined via pullback of L_X on X. This defines an extension of L to a line bundle L_1 on $T_s \times B(b) \times X$, where $B(b)$ is the closed ball of radius the outer radius of T_s.

Applying the induction hypothesis once again, we see that

$$\mathrm{Pic}(T_s^r \times D(b) \times X) = \mathrm{Pic}(D(b) \times X) = \mathrm{Pic}(X)$$

by Lemma 4.6. Thus L_1 is the pullback of L_X. But then the same has to be true for its restriction to $T_s^r \times T_s \times X$. This proves the claim. □

Proof of Proposition 4.5. The Čech-to-sheaf sequence of the standard cover $\mathfrak{U} = (T_s)_{s \in \mathbb{N}}$ of T induces a short exact sequence

$$0 \to \check{H}^1(\mathfrak{U}, \mathcal{O}^\times) \to H^1(\widetilde{T} \times U, \mathcal{O}^\times) \to \check{H}^0(\widetilde{\mathfrak{U}}, H^1(-, \mathcal{O}^\times)) \to 0.$$

Exactly like in the proof of [9, Lemma 6.3.2 and Thm 6.3.3], one sees that the first term is trivial. The last term is equal to $\mathrm{Pic}(X)$ by Lemma 4.8. □

Corollary 4.9. *Let X be the generic fibre of a smooth formal scheme over $\mathcal{O}_K$ and let $\pi : E \to X$ be a T-torsor over X. Then the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(E)$ is surjective.*

Proof. By [15, Lemma 3.6], any T-torsor on X is trivial locally in the Zariski-topology. It follows from this and Proposition 4.5 that the Leray sequence of $\pi : E_{\mathrm{an}} \to X_{\mathrm{Zar}}$ induces an isomorphism

$$\mathrm{Pic}(E) = H^1_{\mathrm{Zar}}(X, \pi_* \mathcal{O}^\times).$$

On the other hand, we have a short exact sequence $\mathcal{O}^\times \to \pi_* \mathcal{O}^\times \to M$ on X_{Zar}. The result follows since $H^1_{\mathrm{Zar}}(X, M) = H^1(\overline{X}, \mathbb{Z}) \otimes M = 0$. □

4.3. $\overline{\mathcal{O}}^\times$**-torsors on universal covers of relative tori.** The goal of this section is to prove an analogue of Proposition 4.5 for $\mathcal{O}^\times[\frac{1}{p}]$-torsor on perfectoid tori.

Proposition 4.10. *Let $U \sim \varprojlim_{i\in I} U_i$ be an affinoid perfectoid tilde-limit of a cofiltered inverse system of generic fibres of smooth affine formal schemes over $\mathcal{O}_K$. Then specialisation at 1 defines isomorphisms*

$$H^1(\widetilde{T}\times U, \overline{\mathcal{O}}^\times) = H^1(U, \mathcal{O}^\times)[\tfrac{1}{p}].$$

Proof. We draw on arguments in the rigid case from [13, §2] and the proof of [9, Lemma 4.7.3 and Thm 6.3.3]. We start by following the latter:

The standard cover $\widetilde{\mathfrak{U}} = (\widetilde{T}_s)_{s\in\mathbb{N}}$ of $\widetilde{T}$ gives a left-exact sequence

$$0 \to \check{H}^1(\widetilde{\mathfrak{U}}, \overline{\mathcal{O}}^\times) \to H^1(\widetilde{T}\times U, \overline{\mathcal{O}}^\times) \to \check{H}^0(\widetilde{\mathfrak{U}}, H^1(-, \overline{\mathcal{O}}^\times)).$$

The first term is easy to control:

Claim 4.11. $\check{H}^1(\widetilde{\mathfrak{U}}, \overline{\mathcal{O}}^\times) = 1.$

Proof. The cover $\widetilde{\mathfrak{U}}$ is an increasing sequence of affinoid perfectoids $\widetilde{T}_s \times U$, for which by Lemma 4.3 we have

$$\overline{\mathcal{O}}^\times(\widetilde{T}_s\times U) = \underline{M^\vee}[\tfrac{1}{p}](U)\times \overline{\mathcal{O}}^\times(R^\times).$$

Since this is independent of s, this shows that $\check{H}^1(\widetilde{\mathfrak{U}}, \overline{\mathcal{O}}^\times) = 0$. □

We are left to see that the right term can be identified with $H^1(U, \mathcal{O}^\times[\frac{1}{p}])$.

Lemma 4.12. *Let $s\in\mathbb{N}$. Then in the situation of Proposition 4.10, the pullback map*

$$\mathrm{Pic}_{\mathrm{an}}(U) \to \mathrm{Pic}_{\mathrm{an}}(\widetilde{T}_s\times U)$$

is an isomorphism. In particular, $\check{H}^0(\widetilde{\mathfrak{U}}, H^1(-, \overline{\mathcal{O}}^\times) = H^1(U, \overline{\mathcal{O}}^\times)$.

Proof. Rescaling defines an isomorphism of rigid spaces $T_s^{(n)} \cong T_s$. Therefore, we can deduce from Lemma 4.8 that for every $i\in I$, we have a natural isomorphism

$$\mathrm{Pic}(T_s^{(n)}\times U_i) = \mathrm{Pic}(U_i)$$

By [10, Corollary 5.4.42], since the U_i are all affinoid, this shows that

$$\mathrm{Pic}(\widetilde{T}_s\times U) = \varinjlim_{n,i}\mathrm{Pic}(T_s^{(n)}\times U_i) = \varinjlim_{i}\mathrm{Pic}(U_i) = \mathrm{Pic}(U).$$

This shows the first part. For the second, we use that all spaces appearing in the lemma are affinoid. Hence [15, Lemma 2.14] and the first part imply that also

$$H^1_{\mathrm{an}}(U, \overline{\mathcal{O}}^\times) \to H^1_{\mathrm{an}}(\widetilde{T}_s\times U, \overline{\mathcal{O}}^\times)$$

is an isomorphism. But by definition, we have $\check{H}^0(\widetilde{\mathfrak{U}}, H^1(-, \overline{\mathcal{O}}^\times)) = \varprojlim H^1(\widetilde{T}_s\times\widetilde{U}, \overline{\mathcal{O}}^\times))$. □

This finishes the proof of Proposition 4.10. □

4.4. $\mathrm{Pic}(\widetilde{T})$ **is huge.** We are now ready to compute $\mathrm{Pic}(\widetilde{T})$. For this we need two more definitions:

Definition 4.13. We denote by $\mathcal{O}_K[[X^{N[\frac{1}{p}]}]]$ the topological $\mathcal{O}_K$-module of formal sums

$$\mathcal{O}_K[[X^{N[\frac{1}{p}]}]] = \left\{ \sum_{m\in N[\frac{1}{p}]} a_m X^m \middle| a_m \in \mathcal{O}_K \right\} \cong \mathcal{O}_K^{N[\frac{1}{p}]}$$

For an element g of this module, we write $g[m]$ for the m-th coefficient.

Recall that for any $s \in \mathbb{N}$ we set

$$\widetilde{T}_s := \widetilde{T}(|\pi|^s \leq |X_i| \leq |\pi|^{-s} \text{ for } i = 1, \dots, r).$$

We note that we can think of any element $f \in \mathcal{O}^+(\widetilde{T}_s)$ as being a formal sum

$$\sum_{n \in N[\frac{1}{p}]} a_n X^n \in \mathcal{O}_K[[X^{N[\frac{1}{p}]}]].$$

with the condition that $a_n \in \pi^{s|n|}\mathcal{O}_K$ and $|a_n| \cdot |\pi|^{-s|n|} \to 0$ for $|n| \to \infty$, plus the additional condition that for any ϵ, there is $k \in \mathbb{N}$ such that $|a_m| \cdot |\pi|^{s|m|} < \epsilon$ for all $m \in N[\frac{1}{p}]$ with

$$v_p(m) := \min_{i \in m}(v_p(i)) \leq -k$$

This additional convergence condition in the perfectoid case is crucial: it is what ultimately gives rise to additional line bundles.

In the following, we work inside the ambient module $\mathcal{O}_K[[X^{N[\frac{1}{p}]}]]$ and in particular we shall tacitly pass from $f \in \mathcal{O}^+(\widetilde{T}_s)$ to its restriction in $\mathcal{O}^+(\widetilde{T}_{s'})$ for any $s' \leq s$ without distinguishing the two in the notation.

Definition 4.14. Let $\mathcal{O}^\star(\widetilde{T}_s)$ be the set of elements $g \in 1+\mathfrak{m}\mathcal{O}^+(\widetilde{T}_s)$ with constant coefficient $g[0] = 1$. The inclusion $\mathcal{O}^\star(\widetilde{T}_s) \subseteq 1 + \mathfrak{m}\mathcal{O}^+(\widetilde{T}_s)$ has a splitting given by sending f to

$$f^\star := f \cdot f[0]^{-1}.$$

Remark 4.15. We note that $\mathcal{O}^\star(\widetilde{T}_s)$ is not closed under the group action of $\mathcal{O}^\star(\widetilde{T}_s)$. Instead, we think of it as a set of representatives for the quotient group $(1 + \mathfrak{m}\mathcal{O}^+(\widetilde{T}_s))/(1 + \mathfrak{m}\mathcal{O}_K)$ of power series up to scaling, which induces on $\mathcal{O}^\star(\widetilde{T}_s)$ the group structure given by sending

$$(f, g) \mapsto f \star g := (f \cdot g)^\star.$$

Another natural set of representatives is given by power series f with $f(1, \dots, 1) = 1$. This has the advantage that it does form a subgroup. But in contrast to the rigid setting, we find it more convenient in the perfectoid setup to work with $\mathcal{O}^\star$ since this interacts better with convergence: If a sequence $(f_s \in \mathcal{O}^\times(\widetilde{T}_s))_{s\in\mathbb{N}}$ converges at each coefficient, then so does $f_s^\star$. The analogue is not true with respect to the other set of representatives, since $(f_s(1, \dots, 1))_{s\in\mathbb{N}}$ need not converge in the perfectoid setup.

Definition 4.16. Let $r \in \mathbb{N}$ and let $(f_s)_{s\geq r}$ be a sequence with $f_s \in \mathcal{O}^\star(\widetilde{T}_s)$. We write

$$\textstyle\prod^\star_{s\geq r} f_s$$

for the sequence of normalised partial sums $g_n := (\prod^n_{s\geq r} f_s)^\star$. We say that $\prod^\star_{s\geq r} f_s$ converges coefficient-wise if the sequence of coefficient $(g_n[m])_{n\geq r}$ converges to an element $g_\infty[m] \in \mathcal{O}_K$ for all $m \in \mathbb{Z}[\frac{1}{p}]$. We also write $\prod^\star_{s\geq r} f_s$ for the limiting power series g_∞ in $\mathcal{O}_K[[X^N]]$.

Theorem 4.17. *The topological group* $\mathrm{Pic}(\widetilde{T})$ *is isomorphic to*

$$\Big\{f \in \prod_{s\in\mathbb{N}} \mathcal{O}^\star(\widetilde{T}_s)\Big\}\Big/\Big\{f \text{ s.t. } \forall r \in \mathbb{N} : \textstyle\prod^\star_{s\geq r} f_s \text{ converges coefficient-wise with limit} \in \mathcal{O}^\times(\widetilde{T}_r)\Big\}.$$

The map is given by sending f to the line bundle that is trivial on each $\widetilde{T}_s$, with glueing data given by $f_s \in \mathcal{O}^\times(\widetilde{T}_s)$. The displayed group is non-trivial and torsionfree: For example,

$$(f_s := 1 + p^{\frac{s+1}{p^s}} X^{\frac{1}{p^s}})_{s\in\mathbb{N}}$$

defines an element for which $\prod^\star_{s\geq 1} f_s$ converges coefficient-wise, but with limit $\notin \mathcal{O}^\times(\widetilde{T}_1)$.

Corollary 4.18. *The Picard group of the punctured perfectoid unit disc* $\mathrm{Spa}(K\langle X^{1/p^\infty}\rangle)\backslash\{0\}$ *is non-trivial. The Picard group of the perfectoid affine line* $\mathbb{A}^{1,\mathrm{perf}}$ *is non-trivial.*

Remark 4.19. Lazard has proved [22, Proposition 6] that since K is not spherically complete, there is $r \in \mathbb{R}$ for which the open rigid unit disc of radius r has non-trivial line bundles (in particular, if the value group of K is $\mathbb{R}$, this holds for the open unit disc).

In this sense, the situation in the perfectoid case is closer to that of the open unit disc than to that of the rigid torus.

Remark 4.20. In contrast, a similar computation as the above shows that

$$H^1(\widetilde{T} \times Z, \mathcal{O}) = 0$$

for any affinoid perfectoid space Z, which might be regarded special case of a perfectoid analogue of Kiehl's vanishing of coherent cohomology on Stein spaces [21, Satz 2.4]. In characteristic 0, one can use this and the next section to see that

$$H^1(\widetilde{T}, \mathcal{O}^\times[\tfrac{1}{p}]) = 0.$$

This is no contradiction to Theorem 4.17 since $\widetilde{T}$ is not quasi-compact.

Remark 4.21. By the same strategy of proof one can show that

$$H^1(\widetilde{T}, \mathcal{O}^+) \overset{a}{=} \prod_{s\in\mathbb{N}} \mathcal{O}_0^+(\widetilde{T}_s)/\{f \in \prod_{s\in\mathbb{N}} \mathcal{O}_0^+(\widetilde{T}_s) \mid \forall s : \sum_{n\geq s} f_n \text{ contained in } \mathcal{O}^+(\widetilde{T}_s)\},$$

where $\mathcal{O}_0^+(T_s) \subseteq \mathcal{O}^+(T_s)$ is the subset with constant coefficient 0. Moreover, one can show that in characteristic 0, the exponential map defines an injection $H^1(\widetilde{T}, \mathcal{O}^+) \hookrightarrow H^1(\widetilde{T}, \mathcal{O}^\times)$. This gives an easier way of seeing that $\mathrm{Pic}(\widetilde{T}) \neq 1$ (but doesn't compute all of $\mathrm{Pic}(\widetilde{T})$).

For the proof of Theorem 4.17, we need:

Lemma 4.22. *Let $g \in \mathcal{O}^\star(\widetilde{T}_r)$ and let $h_n \in \mathcal{O}^\star(\widetilde{T}_r)$ be a sequence that converges coefficient-wise with limit $h \in \mathcal{O}^\star(\widetilde{T}_r)$. Then $g\star h_n$ converges coefficient-wise with limit $g\star h \in \mathcal{O}^\star(\widetilde{T}_r)$.*

Proof. It suffices to prove that $g \cdot h_n$ in $\mathcal{O}^\times(\widetilde{T}_r)$ converges coefficient-wise to $g \cdot h$. After rescaling, we may assume that $r = 1$. Write $g = \sum a_m \pi^{|m|} X^m$ and $h_n = \sum b_m^{(n)} \pi^{|m|} X^m \in \mathcal{O}^\star(\widetilde{T}_r)$ with limits $b_m \in \mathcal{O}_K$. Then

$$g \cdot h_n = \sum_{m \in N[\frac{1}{p}]} \left(\sum_{i+j=m} a_i b_j^{(n)} \pi^{(|i|+|j|)} \right) X^m.$$

Let $m \in N[\frac{1}{p}]$, we consider the m-th coefficient $(g \cdot h_n)[m]$. Let $l \in N$. As $g \in \mathcal{O}^\star(\widetilde{T}_r)$, there is a finite set J such that $a_j \in \pi^l \mathcal{O}_K$ unless $j \in J$. We then have

$$(g \cdot h_n)[m] \equiv \sum_{\substack{i+j=m \\ j\in J}} a_i b_j^{(n)} \pi^{(|i|+|j|)} \bmod \pi^l.$$

Since this is a finite sum, we can now find $n \gg 0$ such that $b_j^{(n)} \equiv b_j \bmod \pi^l$ for all $j \in J$. Then we have

$$(g \cdot h_n)[m] \equiv g \cdot h \bmod \pi^l \mathcal{O}_K.$$

For $l \to \infty$, this shows the desired coefficient-wise convergence. □

Proof of Theorem 4.17. We start by following the proof of [9, Theorem 6.3.3], even though we end up getting a different result: We use the cover $\mathfrak{U} = (\widetilde{T}_s)_{s\in\mathbb{N}}$, for which the Čech-to-sheaf spectral sequence gives an exact sequence

$$0 \to \check{H}^1(\mathfrak{U}, \mathcal{O}^\times) \to H^1(\widetilde{T}, \mathcal{O}^\times) \to \check{H}^0(\mathfrak{U}, H^1(-, \mathcal{O}^\times)).$$

The right term vanishes: Indeed, by [9, Lemma 6.3.1], we have $\mathrm{Pic}(T_s^{(n)}) = 1$ for all $n \in \mathbb{N}$, where $T^{(n)}$ is the torus with character group $\frac{1}{p^n}N$. The desired vanishing follows in the limit from [10, Corollary 5.4.42] since the $T_s^{(n)}$ are affinoid.

It thus suffices to consider the left term. Since $\mathfrak{U}$ is an increasing union, the resulting truncated Čech complex is quasi-isomorphic to the complex concentrated in degrees $[0,1]$

$$\prod_{s\in\mathbb{N}}\mathcal{O}^\times(\widetilde{T}_s)\xrightarrow{h}\prod_{s\in\mathbb{N}}\mathcal{O}^\times(\widetilde{T}_s),$$

where h is the map that sends

$$(f_s)\mapsto(f_sf_{s+1}^{-1}).$$

The kernel of h is $\mathcal{O}^\times(\widetilde{T})=K^\times\times N[\frac{1}{p}]$ by Lemma 4.3; the cokernel computes $\check{H}^1(\mathfrak{U},\mathcal{O}^\times)$.

Recall from Lemma 4.3.4 that we have a short exact sequence

$$1\to K^\times\times N[\tfrac{1}{p}]\to\mathcal{O}^\times(\widetilde{T}_s)\to(1+\mathfrak{m}\mathcal{O}^+(\widetilde{T}_s))/(1+\mathfrak{m}\mathcal{O}_K)\to 0.$$

The first term is constant throughout the inverse system over s. By applying h to the short exact sequence, we conclude that this first term does not contribute to the cokernel, and we are therefore reduced to considering the cokernel of h in $(1+\mathfrak{m}\mathcal{O}^+(\widetilde{T}_s))/(1+\mathfrak{m}\mathcal{O}_K)$. Recall that the inclusion

$$\mathcal{O}^\star(\widetilde{T}_s)\to(1+\mathfrak{m}\mathcal{O}^+(\widetilde{T}_s))/(1+\mathfrak{m}\mathcal{O}_K),$$

defines a bijection which endows the left hand side with a group action given by sending

$$(f,g)\mapsto f\star g:=(f\cdot g)^\star.$$

It thus suffices to compute the cokernel of

$$\prod_{s\in\mathbb{N}}\mathcal{O}^\star(\widetilde{T}_s)\xrightarrow{h}\prod_{s\in\mathbb{N}}\mathcal{O}^\star(\widetilde{T}_s).$$

We need to see that the image of this map is precisely

$$\operatorname{im}h=\Big\{f\text{ s.t. }\forall r:\textstyle\prod^\star_{s\geq r}f_s\text{ converges coefficient-wise with limit}\in\mathcal{O}^\times(\widetilde{T}_r)\Big\}.$$

We first prove the inclusion "$\subseteq$". Let thus $g\in\prod_{s\in\mathbb{N}}\mathcal{O}^\star(\widetilde{T}_s)$ and set

$$f_s:=g_s\star g_{s+1}^{-1}.$$

Then for any $k\geq r+1$, we have

$$\textstyle\prod_{s=r}^{k-1\star}f_s=g_r\star g_k^{-1}\in\mathcal{O}^\star(\widetilde{T}_s).$$

We have to prove that for $k\to\infty$, we have coefficient-wise convergence

$$g_r\star g_k^{-1}\to g_r.$$

By Lemma 4.22, it suffices to see that $(g_k^{-1})^\star\to 1$. But $(g_k^{-1})^\star\in\mathcal{O}^\star(\widetilde{T}_s)$ implies that for each $m\in N[\frac{1}{p}]$, we have $(g_k^{-1})^\star[m]\in\pi^{|m|k}\mathcal{O}_K$, which gives the desired coefficient-wise convergence for $k\to\infty$. This proves the containment "$\subseteq$".

To prove the converse direction, let $(f_s)_{s\in\mathbb{N}}$ be such that each $\prod^\star_{s\geq r}f_s$ converges coefficient-wise with limit

$$g_r:=\textstyle\prod^\star_{s\geq r}f_s\in\mathcal{O}^\times(\widetilde{T}_r).$$

Then we automatically have $g_r\in\mathcal{O}^\star(\widetilde{T}_r)$, since the condition for $g_r\in\mathcal{O}^\times(\widetilde{T}_r)$ to be in this subgroup is that $a_m\in\mathfrak{m}\pi^{s|m|}$ and $a_0=1$, which is preserved under coefficient-wise convergence. We claim that

$$g_r\star g_{r+1}^{-1}=f_r.$$

Equivalently, we need to see that

$$\textstyle\prod^\star_{s\geq r}f_s=f_r\star\prod^\star_{s\geq r+1}f_s.$$

In terms of the normalised partial sums $h_n:=\prod^{n\star}_{s\geq r+1}f_s$, this means that

$$f_r\star h_n\to f_r\star h,$$

which is true by Lemma 4.22.

This shows that $(g_s)_{s\in\mathbb{N}}\in\prod\mathcal{O}^\star(\widetilde{T}_s)$ is sent by h to $(g_s\star g_{s+1}^{-1})_{s\in\mathbb{N}}=(f_s)_{s\in\mathbb{N}}$, proving that f is in the image of h. This finishes the proof of the desired description of $\operatorname{im}h$. □

5. Picard groups of universal covers

The final goal of this article is to answer Question 1.1 when A is an abeloid variety, i.e. we will describe $\mathrm{Pic}(\widetilde{A})$. We already know the answer in the case of good reduction:

Theorem 5.1 ([15, Theorem 4.1, Corollary 4.10])**.** *Let B be an abeloid variety over K of good reduction $\overline{B}$ over k. Then there is a natural isomorphism*

$$\mathrm{Pic}(\widetilde{B}) = \mathrm{Pic}(\overline{B}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

On identity components, this identifies the pullback map $\mathrm{Pic}(B) \to \mathrm{Pic}(\widetilde{B})$ with the reduction

$$B^\vee(K) \to \overline{B}^\vee(k) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the kernel of $\mathrm{Pic}(B) \to \mathrm{Pic}(\widetilde{B})$ is the topological torsion subgroup $\mathrm{Pic}(B)^{\mathrm{tt}}$.

This gives an example where the map from Question 1.1 is neither injective nor surjective, because the Néron–Severi group of $\overline{B}$ can be larger than that of B.

The goal of this section is to deduce from Theorem 5.1 a description of the kernel of $\mathrm{Pic}(A) \to \mathrm{Pic}(\widetilde{A})$ for any abeloid variety. In the next section, we will deal with the cokernel, which will be more difficult.

5.1. The group $\mathrm{Pic}^0(\widetilde{A})$ in general.

Let K be any perfectoid field over $\mathbb{Z}_p$ and let A be an abeloid over K with its universal cover $\widetilde{A} = \varprojlim_{[N]} \to A$. In this section, we answer the first part of Question 1.1, namely we determine the kernel of

$$\mathrm{Pic}(A) \to \mathrm{Pic}(\widetilde{A}). \tag{4}$$

We do this by describing $\mathrm{Pic}^0_{\mathrm{an}}(\widetilde{A}) = \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m)$ in the category of sheaves on $\mathrm{Perf}_{K,\mathrm{an}}$.

Definition 5.2. We denote by $\mathrm{Pic}^0(\widetilde{A}) \subseteq \mathrm{Pic}(\widetilde{A})$ the subgroup of translation invariant line bundles L, i.e. those that satisfy $x^*L \cong L$ for all $x \in \widetilde{A}(K)$.

Lemma 5.3. *We have $\mathrm{Pic}^0(\widetilde{A}) = \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m)$.*

This follows immediately from the Breen–Deligne sequence:

Lemma 5.4. *Let F be an abelian sheaf in $\mathrm{Perf}_{K,v}$ such that $\mathrm{Map}(F, \mathbb{G}_m) = K^\times$. Then for $\tau = \mathrm{an}$ or $\tau = v$, there is a natural left-exact sequence*

$$0 \to \mathrm{Ext}^1_\tau(F, \mathbb{G}_m) \to H^1_\tau(F, \mathbb{G}_m) \xrightarrow{[m]^* - [\pi_1]^* - [\pi_2]^*} H^1_\tau(F \times F, \mathbb{G}_m).$$

Proof. The Breen–Deligne spectral sequence E of [2, §2.1] gives an exact sequence

$$0 \to E_2^{10} \to \mathrm{Ext}^1_{\mathrm{an}}(F, \mathbb{G}_m) \to \ker([m]^* - [\pi_1]^* - [\pi_2]^*) \to E_2^{20}, \tag{5}$$

where $E_2^{\bullet 0}$ is the cohomology of a complex whose terms are finite products of $\mathrm{Map}(F, \mathbb{G}_m)$. In particular, if this is $\mathrm{Map}(\mathrm{Spa}(K), \mathbb{G}_m)$, then these terms are the same as the ones appearing in the sequence for $F = \mathrm{Spa}(K)$, which shows that they must vanish. □

We thus obtain the following Corollary to Theorem 5.1:

Corollary 5.5. *Let B be an abeloid variety of good reduction. Then*

$$\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{B}, \mathbb{G}_m) = \overline{B}^\vee(k) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. By the Breen–Deligne sequence of Lemma 5.4, we have

$$\mathrm{Ext}^1(\widetilde{B}, \mathbb{G}_m) = \ker\big(\mathrm{Pic}(\widetilde{B}) \to \mathrm{Pic}(\widetilde{B} \times \widetilde{B})\big) = \ker\big(\mathrm{Pic}(\overline{B}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}(\overline{B} \times \overline{B}) \otimes_{\mathbb{Z}} \mathbb{Q}\big),$$

which is precisely $\mathrm{Ext}^1_{\mathrm{Zar}}(\overline{B}, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q} = \overline{B}^\vee(k) \otimes_{\mathbb{Z}} \mathbb{Q}$. □

We can use this to describe $\mathrm{Pic}^0(\widetilde{A})$ in the general case as follows, using Definition 2.3:

Theorem 5.6. *Let A be an abeloid variety over K. Let $A^\vee$ be its dual. Then for $\tau = \mathrm{an}, v$,*

$$(i) \qquad \mathrm{Ext}^1_\tau(\widetilde{A}, \mathbb{G}_m) = \big(A^\vee(K)/\widehat{A}^\vee(K)\big)[\tfrac{1}{p}],$$

$$(ii) \qquad \mathrm{Ext}^1_\tau(\widetilde{\widetilde{A}}, \mathbb{G}_m) = \big(A^\vee(K)/A^{\vee\mathrm{tt}}(K)\big) \otimes \mathbb{Q}.$$

The proof will be completed in two steps in the following subsections. But before, we give our main Corollary: We can now answer the first part of Question 1.1 in great generality, extending from the case of algebraically closed K over $\mathbb{Q}_p$ treated in [14]. We first note:

Corollary 5.7. *We have a left-exact sequences*

$$
\begin{array}{ll}
(i) & 0 \longrightarrow \widehat{A}^\vee(K) \longrightarrow \mathrm{Pic}_{\mathrm{an}}(A) \longrightarrow \mathrm{Pic}_{\mathrm{an}}(\widetilde{A}_p), \\
(i)' & 0 \longrightarrow \widehat{A}^\vee(K) \longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(A, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}_p, \mathbb{G}_m), \\
(ii) & 0 \longrightarrow A^\vee(K)^{\mathrm{tt}} \longrightarrow \mathrm{Pic}_{\mathrm{an}}(A) \longrightarrow \mathrm{Pic}_{\mathrm{an}}(\widetilde{A}), \\
(ii)' & 0 \longrightarrow A^\vee(K)^{\mathrm{tt}} \longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(A, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m).
\end{array}
$$

Corollary 5.8. *Let K be any non-archimedean field over $\mathbb{Z}_p$ and let X be a smooth proper rigid space over K for which $\mathbf{Pic}_X$ is represented by a rigid group whose identity component is abeloid. Then every line bundle on X that is topologically torsion in $\mathbf{Pic}_X(K)$ is trivialised by a pro-finite cover of X, and even by a pro-finite-étale cover if* $\mathrm{char}\, K = 0$.

Proof. Let $L \in \mathbf{Pic}_X(K)$ be topologically torsion. Any line bundle whose image in the Néron–Severi group is torsion is trivialised by a finite étale cover, so we can without loss of generality assume that $L \in \mathrm{Pic}^0(X)$.

Let A be the dual abeloid variety of $\mathbf{Pic}^0_X$. Then by [12, §4], there is a natural map $X \to A$ that satisfies the universal property of the Albanese variety. By the assumption hat $\mathbf{Pic}^0_X$ is abeloid, this map induces an isomorphism $\mathbf{Pic}^0_A \xrightarrow{\sim} \mathbf{Pic}^0_X$. In particular, L comes via pullback from a topologically torsion line bundle on A. By the previous Corollary, it thus becomes trivial on $\widetilde{A}$, hence it is trivial on the pro-finite cover $\widetilde{A} \times_A X \to X$. □

Remark 5.9. In general, it is expected that $\mathbf{Pic}^0_X$ is a semi-abeloid variety (due to [13]), but in many cases it is known to be abeloid, for example if X has projective reduction [23].

Proof of Corollary 5.7. Let $N := \ker(H^1_{\mathrm{an}}(A, \mathbb{G}_m) \to H^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m))$, then by Cartan–Leray and Theorem 5.6, we have a commutative square of left-exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & H^1_{\mathrm{an}}(A, \mathbb{G}_m) & \longrightarrow & H^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m) \\
0 & \longrightarrow & A^\vee(K) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{an}}(A, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}, \mathbb{G}_m) \\
0 & \longrightarrow & \mathrm{Hom}(TA, \mathbb{G}_m) & \longrightarrow & H^1_v(A, \mathbb{G}_m) & \longrightarrow & H^1_v(\widetilde{A}, \mathbb{G}_m) \\
0 & \longrightarrow & \mathrm{Hom}(TA, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_v(A, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_v(\widetilde{A}, \mathbb{G}_m).
\end{array}
$$

Since the second vertical arrow on the back is injective, the map $N \to \mathrm{Hom}(TA, \mathbb{G}_m)$ is injective. The second diagonal square is Cartesian, this follows from comparing Lemma 5.4 for the analytic and v-topology. Since the bottom left arrow is an equality, this shows that N already lands in $\mathrm{Ext}^1_{\mathrm{an}}(A, \mathbb{G}_m)$. Since the rightmost diagonal arrows are inclusions, the top left diagram is Cartesian as well. This shows that N already lies in $A^\vee(K)$, thus showing that the morphism $A^\vee(K) \to N$ is an isomorphism. □

5.2. **The algebraically closed case.** Assume that K is algebraically closed. In this situation, there exists a Raynaud uniformisation of A. We briefly recall what this is, and refer to [26, §6] for details: One can associate to A a short exact sequence of rigid groups in the analytic topology

$$0 \to T \to E \to B \to 0$$

where B is an abeloid variety with good reduction and T is a rigid torus of some rank r, and a lattice $\mathbb{Z}^r \cong M \subseteq E$ such that there is a short exact sequence in the analytic topology

$$0 \to M \to E \to A \to 0.$$

There is an analogous analytic uniformisation for $\widetilde{A}$:

Theorem 5.10 ([3, Theorem 5.6]). *(1) The covers* $\widetilde{T}_p := \varprojlim_{[p]} T$, $\widetilde{E}_p := \varprojlim_{[p]} E$ *and* $\widetilde{B}_p$ *are all represented by perfectoid spaces. They fit into an exact sequence on* $\mathrm{Perf}_{K,\mathrm{an}}$

$$0 \to \widetilde{T}_p \to \widetilde{E}_p \to \widetilde{B}_p \to 0.$$

(2) Let $M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *and set* $X := \widetilde{E}_p \times M_p$. *Then any choice of lift* $M \to \widetilde{E}_p$ *of* $M \to E$ *induces a pullback diagram of short exact sequences of adic groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & \widetilde{A}_p & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{6}$$

in the analytic topology. Here $M \to X$ *is the anti-diagonal embedding.*

We now use this to reduce Theorem 5.6 to the case of good reduction. For this we need:

Lemma 5.11. *(1)* $\mathrm{Ext}^1_v(\mathbb{Z}_p, \mathbb{G}_m) = \mathrm{Ext}^1_{\mathrm{an}}(\mathbb{Z}_p, \mathbb{G}_m) = 0.$
(2) $\mathrm{Ext}^1_{\mathrm{an}}(\mathbb{Z}_p, \widehat{\mathbb{G}}_m) = \mathrm{Ext}^1_v(\mathbb{Z}_p, \widehat{\mathbb{G}}_m) = 0.$

Proof. We begin with part 2: Let $1 \to \widehat{\mathbb{G}}_m \to \mathcal{F} \xrightarrow{\pi} \mathbb{Z}_p \to 0$ be an extension. Then since $\mathcal{F}$ is locally in the analytic topology on $\mathbb{Z}_p$ of the form $\widehat{\mathbb{G}}_m \times U$ where $U \subseteq \mathbb{Z}_p$, we see that $\mathcal{F}$ is itself represented by an adic group.

We now pass to K-points: the fibre of $\mathcal{F}(K)$ over $1 \in \mathbb{Z}_p(K) = \mathbb{Z}_p$ is isomorphic to $\widehat{\mathbb{G}}_m$, and thus the fibre of $\widehat{\mathcal{F}}(K)$ over 1 is non-empty. We conclude that there is a group homomorphism $\mathbb{Z}_p \to \mathcal{F}$ splitting π.

To deduce part 1, it suffices to see that the Breen–Deligne complex $E_1^{\bullet 0}$ ([16, §A]) satisfies

$$E_2^{10}(\mathbb{Z}_p, \mathbb{G}_m) = 0.$$

We recall that $E_1^{\bullet 0}$ consists of terms of the form $\oplus_i H^0(\mathbb{Z}_p^{n_i}, \mathbb{G}_m) = \oplus_i \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^{n_i}, K^\times)$. Considering the short exact sequence

$$\mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^{n_i}, 1+\mathfrak{m}) \to \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^{n_i}, K^\times) \to \mathrm{Map}_{\mathrm{lc}}(\mathbb{Z}_p^{n_i}, K^\times/(1+\mathfrak{m})),$$

we see that we get a short exact sequence

$$0 \to E_1^{\bullet 0}(\mathbb{Z}_p, \widehat{\mathbb{G}}_m) \to E_1^{\bullet 0}(\mathbb{Z}_p, \mathbb{G}_m) \to \varinjlim_n E_1^{\bullet 0}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m/\widehat{\mathbb{G}}_m) \to 0.$$

But we have $\mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m/\widehat{\mathbb{G}}_m) = 0$ since the first entry is p^n-torsion while the latter is p-divisible. This shows that $E_2^1(\mathbb{Z}_p, \mathbb{G}_m) = E_2^1(\mathbb{Z}, \widehat{\mathbb{G}}_m) = 0$ by the second part. ☐

Lemma 5.12. $\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{T}, \mathbb{G}_m) = \mathrm{Ext}^1_v(\widetilde{T}, \mathbb{G}_m) = 0.$

Proof. It suffices by linearity to consider $T = \mathbb{G}_m$. The first term of the Breen–Deligne sequence (5) does not depend on the topology, and it agrees with the term computing $\mathrm{Ext}^1_{\mathrm{an}}(\mathbb{G}_m, \mathbb{G}_m)$ after inverting p. But this is $= 1$ as we can see from any Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$. We deduce that the map $\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{T}, \mathbb{G}_m) \hookrightarrow \mathrm{Ext}^1_v(\widetilde{T}, \mathbb{G}_m)$ is injective, so it suffices to consider the v-topology. Via tilting, we further reduce to char $K = 0$.

Second, we deduce that the Breen–Deligne spectral sequence yields an exact sequence

$$0 \to \mathrm{Ext}^1_v(\mathbb{G}_m, \mathbb{G}_m) \to H^1_v(\mathbb{G}_m, \mathbb{G}_m) \xrightarrow{d} H^1_v(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m),$$

where $d = [m]^* - [\pi_1]^* - [\pi_2]^*$. By [17, Theorem 1.3.1], this fits in a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}^1_v(\mathbb{G}_m, \mathbb{G}_m) & \longrightarrow & \mathrm{Pic}_v(\mathbb{G}_m) & \xrightarrow{d} & \mathrm{Pic}_v(\mathbb{G}_m \times \mathbb{G}_m) \\ & & \vdots & & \downarrow{\scriptstyle \mathrm{HT}\log}\ \wr & & \downarrow \\ 0 & \longrightarrow & \ker d & \longrightarrow & \Omega^1(\mathbb{G}_m)(-1) & \xrightarrow{d} & \Omega^1(\mathbb{G}_m \times \mathbb{G}_m)(-1), \end{array}$$

where the second vertical map is an isomorphism. It follows that $\mathrm{Ext}^1_v(\mathbb{G}_m,\mathbb{G}_m)=K(-1)$ are precisely the translation invariant differentials on T generated by the canonical differential $\frac{dx}{x}$ on $\mathbb{G}_m$. Consider now the long exact sequence of $\mathbb{Z}_p(1)\to\widetilde{\mathbb{G}}_m\to\mathbb{G}_m$

$$\mathrm{Hom}(\mathbb{Z}_p(1),\mathbb{G}_m)\xrightarrow{\log}\mathrm{Ext}^1_v(\mathbb{G}_m,\mathbb{G}_m)\to\mathrm{Ext}^1_v(\widetilde{\mathbb{G}}_m,\mathbb{G}_m)\to\mathrm{Ext}^1_v(\mathbb{Z}_p(1),\mathbb{G}_m).$$

By the above, log is surjective. The last term is $=0$ by the second part of the lemma. □

Proof of Theorem 5.6. We first assume that K is algebraically closed. Applying $\mathrm{Hom}(-,\mathbb{G}_m)$ to (6), we obtain a morphism of long exact sequences

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{Hom}(X,\mathbb{G}_m) & \to & \mathrm{Hom}(M,\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A},\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(X,\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(M,\mathbb{G}_m)=0 \\ & & \uparrow & & \| & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathrm{Hom}(E,\mathbb{G}_m) & \to & \mathrm{Hom}(M,\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(A,\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(E,\mathbb{G}_m) & \to & \mathrm{Ext}^1_{\mathrm{an}}(M,\mathbb{G}_m)=0. \end{array}$$

With our preparations, we can compute all terms in columns 1,2,4 and 5. However, this would a priori only show that $\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A},\mathbb{G}_m)$ and $A(K)/\widehat{A}^\vee(K)$ are both extensions of the same two groups. To simplify the situation, we shall instead use some additional input:

We first reduce to characteristic 0. This is possible since by [20], for any abeloid A' over $K^\flat$ one can find A over K with $\widetilde{A}'_p=\widetilde{A}^\flat_p$ and $A'(K)/\widehat{A}'(K)=A(K)/\widehat{A}(K)$. In characteristic 0, we know by [14] that we have a left-exact sequence

$$0\to A^{\vee\mathrm{tt}}(K)\to\mathrm{Ext}^1_{\mathrm{an}}(A,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A},\mathbb{G}_m).$$

As $\widetilde{A}\to\widetilde{A}_p$ is a T^pA-torsor where $T^pA=\varprojlim_{(p,N)=1}A[N]$, we also have a short exact sequence

$$1\to\mathrm{Hom}(T^pA,\mathbb{G}_m)\to\mathrm{Ext}^1_v(\widetilde{A}_p,\mathbb{G}_m)\to\mathrm{Ext}^1_v(\widetilde{A},\mathbb{G}_m)\to 1. \tag{7}$$

Via the Weil pairing, the first term gets identified with the prime-to-p torsion

$$\mathrm{Hom}(T^pA,\mathbb{G}_m)=\varinjlim_{(N,p)=1}A^\vee[N](K)\subseteq A^\vee(K).$$

Since $A^{\vee\mathrm{tt}}(K)=\widehat{A}^\vee(K)\times\varinjlim_{(N,p)=1}A^\vee[N]$ by [14, Proposition 2.19.2], we deduce that we have an injective map

$$A(K)/\widehat{A}^\vee(K)\hookrightarrow\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{A}_p,\mathbb{G}_m),$$

and we are left to see that this is surjective. This can now be done using the above diagram:

The long exact sequence of $T\to E\to B$ reads

$$\mathrm{Hom}(T,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(B,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(E,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(T,\mathbb{G}_m).$$

The last term vanishes. Let $M^\vee:=\mathrm{Hom}(T,\mathbb{G}_m)$, then we conclude

$$\mathrm{Ext}^1_{\mathrm{an}}(E,\mathbb{G}_m)=B^\vee(K)/M^\vee.$$

Similarly, the sequence of Theorem 5.10.1 induces an exact sequence

$$\mathrm{Hom}(\widetilde{T}_p,\mathbb{G}_m)\xrightarrow{\delta}\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{B}_p,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{E}_p,\mathbb{G}_m)\to\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{T}_p,\mathbb{G}_m).$$

The last term vanishes by Lemma 5.12, as does $\mathrm{Ext}^1(M_p,\mathbb{G}_m)=0$ by Lemma 5.11. Hence

$$\mathrm{Ext}^1_{\mathrm{an}}(X,\mathbb{G}_m)=\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{E}_p,\mathbb{G}_m)=(\overline{B}^\vee(k)/\delta(M^\vee))[\tfrac{1}{p}].$$

Combining these computations, we conclude that the above diagram is of the form

$$\begin{array}{ccccccccc} \dots & \longrightarrow & T^\vee(K) & \longrightarrow & \mathrm{Ext}^1(\widetilde{A},\mathbb{G}_m) & \longrightarrow & (\overline{B}^\vee(k)/\delta(M^\vee))[\tfrac{1}{p}] & \longrightarrow & 0 \\ & & \| & & \uparrow & & \twoheaduparrow & & \\ 0 & \longrightarrow & T^\vee(K) & \longrightarrow & A^\vee(K) & \longrightarrow & B^\vee(K)/M^\vee & \longrightarrow & 0 \end{array}$$

where $T^\vee:=\mathrm{Hom}(M,\mathbb{G}_m)$. The vertical map on the right is surjective since K is algebraically closed. This implies that the middle arrow is surjective, as we wanted to see.

5.3. **Galois descent.** We deduce the case of general perfectoid K by Galois descent: It suffices to prove that

$$\mathrm{Ext}^1_v(\widetilde{A}, \mathbb{G}_m) = (A(K)/\widehat{A}(K)) \otimes \mathbb{Q},$$

the case of $\widetilde{A}_p$ then follows from (7). The case of $\tau = \mathrm{an}$ follows using that $\mathrm{Ext}^1_{\mathrm{an}} \hookrightarrow \mathrm{Ext}^1_v$.

Let C be the completion of an algebraic closure of K and let $G := \mathrm{Gal}(C|K)$ be its Galois group. The Cartan–Leray sequence for $\widetilde{A}_C \to \widetilde{A}_K$ is a left-exact sequence

$$0 \to H^1_{\mathrm{cts}}(G, C^\times) \to \mathrm{Pic}_v(\widetilde{A}_K) = \mathrm{Pic}_v(\widetilde{A}_C)^G.$$

The first term vanishes as $\mathrm{Pic}_v(K) = 1$, so the natural map $\mathrm{Ext}^1_v(\widetilde{A}_K, \mathbb{G}_m) \to \mathrm{Ext}^1_v(\widetilde{A}_C, \mathbb{G}_m)$ is injective. It thus suffices to see that

$$(A^\vee(C)/A^{\vee\mathrm{tt}}(C))^G = \big(A^\vee(K)/A^{\vee\mathrm{tt}}(K)\big) \otimes \mathbb{Q}, \tag{8}$$

since $\varinjlim_{[N]^*} \mathrm{Ext}^1_{\mathrm{an}}(A, \mathbb{G}_m) = A^\vee(K) \otimes \mathbb{Q}$ clearly surjects onto the right hand side.

To prove (8), let L range through finite Galois extensions $L|K$ inside C, then the image of $\varinjlim_{L|K} A^\vee(L) \to A^\vee(C)$ is dense. Since $A^{\vee\mathrm{tt}}(C) \subseteq A^\vee(C)$ is open, we therefore have

$$(A^\vee(C)/\widehat{A}^{\vee\mathrm{tt}}(C))^G = \varinjlim_{L|K} (A^\vee(L)/A^{\vee\mathrm{tt}}(L) \otimes \mathbb{Q})^{\mathrm{Gal}(L|K)}.$$

Since $\mathrm{Gal}(L|K)$ is finite and $A^{\vee\mathrm{tt}}(L) \otimes \mathbb{Q}$ is uniquely divisible, we have

$$H^1(\mathrm{Gal}(L|K), A^{\vee\mathrm{tt}}(L) \otimes \mathbb{Q}) = 0$$

by [30, Proposition (1.6.2)]. This shows that

$$(A^\vee(L)/A^{\vee\mathrm{tt}}(L) \otimes \mathbb{Q})^{\mathrm{Gal}(L|K)} = A^\vee(K)/A^{\vee\mathrm{tt}}(K) \otimes \mathbb{Q}.$$

This proves that (8) holds, which finishes the proof of Theorem 5.6. □

6. Picard groups of universal covers of abeloids

6.1. **Setup.** Let A be an abeloid variety over K. We begin by recalling the main statements about uniformisation of abeloid varieties, due to Raynaud and Bosch–Lütkebohmert: Let r be the rank of $H^1_{\mathrm{an}}(A, \mathbb{Z})$ as a $\mathbb{Z}$-module. Then there exists a unique analytic covering $E \to A$ of A with $H^1(E, \mathbb{Z}) = 0$ such that

$$A = E/M$$

for some lattice $\mathbb{Z}^r \cong M \subseteq E$. Moreover, E is a Raynaud extension, namely there exists a exact of rigid groups which is short exact in the analytic topology

$$0 \to T \to E \xrightarrow{\pi} B \to 0 \tag{9}$$

where T is a torus of rank r and B is an abeloid variety of good reduction. We denote by

$$\phi : M \to B$$

the composition of the inclusion $M \subseteq E$ with the projection $\pi : E \to B$.

Recall now from the duality theory of abeloid varieties that the Picard functor of A is representable and its identity component is an abeloid variety $A^\vee$. Let $A^\vee = E^\vee/M^\vee$ and $T^\vee \to E^\vee \to B^\vee$ be its Raynaud uniformisation. As we will later recall in more detail, there is a canonical identification

$$M^\vee = \mathrm{Hom}(T, \mathbb{G}_m).$$

Moreover, $B^\vee$ is the dual of B, in particular we have

$$\mathrm{Ext}^1_{\mathrm{an}}(B, \mathbb{G}_m) = B^\vee(K).$$

This induces an isomorphism

$$\mathrm{Ext}^1_{\mathrm{an}}(B, T) = \mathrm{Hom}(M^\vee, \mathrm{Ext}^1_{\mathrm{an}}(B, \mathbb{G}_m)) = \mathrm{Hom}(M^\vee, B^\vee(K)),$$

and under this identification, the extension class of E corresponds to the morphism

$$\phi^\vee : M^\vee \to B^\vee(K)$$

associated to $A^\vee$.

6.2. **Recollections: Appell–Humbert for abeloids.** In the classical theory of complex abelian varieties, the description of the Picard variety is achieved by the Appell–Humbert Theorem. We now collect some results of [26, §6.1-§6.3] concerning the analogous computation of $\mathrm{Pic}(A)$ in the rigid-analytic setting. In fact, for later applications to the perfectoid case, we will take a slightly different perspective than [26]. Namely, we wish to reinterpret the construction in terms of the Cartan–Leray spectral sequence for the cover $E \to A$.

Like in complex geometry, the basic idea is to describe line bundles on A in terms of line bundles on E together with descent data. For this we first describe $\mathcal{O}(E)^\times$.

Definition 6.1. Recall that E defines an extension class in $\mathrm{Ext}^1_{\mathrm{an}}(B,T)$ corresponding to a morphism

$$\phi^\vee : M^\vee \to B^\vee(K).$$

Let $M_1^\vee$ be the kernel of $\phi^\vee$, this is a finite free $\mathbb{Z}$-module of rank in between 0 and r.

The following Lemma clarifies the conceptual meaning of M_1, namely it corresponds to the maximal quotient of A that is a totally degenerated abeloid variety:

Lemma 6.2.
1. *We have $M_1^\vee = \mathrm{Hom}(E,\mathbb{G}_m)$. In particular, there is a natural morphism $q : E \to T_1$ to the torus T_1 with character group $M_1^\vee$.*
2. *There is a totally degenerated abeloid A_1 with a surjection $\varphi : A \to A_1$ such that any morphism $A \to A_1'$ into a totally degenerated abeloid admits a unique factorisation through φ. We have $\dim A_1 = \mathrm{rk}_{\mathbb{Z}} M_1^\vee$.*
3. *Assume that the inclusion $M_1^\vee \to M^\vee$ has torsionfree cokernel, so that it corresponds to an extension of tori $T_2 \to T \to T_1$. Let $M_1 := q(M) \subseteq T_1$ and $M_2 := M \cap \ker q$. Then φ is split and A admits a canonical decomposition*

$$A = A_1 \times A_2$$

where $A_1 = T_1/M_1$ and $A_2 = E_2/M_2$ for $E_2 = E \times^T T_1$.

4. *We can always find an isogeny $A' \to A$ such that A' admits a decomposition as in 3.*

Proof.
1. Applying $\mathrm{Hom}(-,\mathbb{G}_m)$ to (16) yields a long exact sequence

$$0 \to \mathrm{Hom}(E,\mathbb{G}_m) \to M^\vee \to \mathrm{Ext}^1(B,\mathbb{G}_m).$$

It follows immediately from the characterisation of $\phi^\vee : M^\vee \to B^\vee(K)$ given in the previous section that the boundary map is identified with $\phi^\vee : M^\vee \to B^\vee$. This shows the first part. In particular, we have a natural map $E \to T_1$ where T_1 has character group $M_1^\vee$.

2. Let $A_1' = T_1'/M_1'$ be any totally degenerate abeloid, then by the universal property of the analytic cover, any morphism $A \to A_1$ lifts to a morphism $\lambda : E \to T_1'$. This induces a factorisation $E \to T_1 \to T_1'$. The fact that λ maps M into M_1' implies that this induces the desired morphism $A_1 \to A'$.
3. If $M_1^\vee \to M^\vee$ has torsionfree cokernel, it is split, hence the map $E \to T_1$ has a splitting $s : T_1 \to E$. The induced map $s : T_1 \to E \to E/M$ has kernel $s^{-1}(M)$. On the other hand, the composition $T_1 \to E/M \to T_1/M_1$ has kernel M_1. It follows that $M_1 = s^{-1}(M)$, hence s sends M_1 into M. This induces the desired splitting $s : T/M_1 \to A/M$. All other statements are then clear.
4. Let $N \in \mathbb{N}$ be large enough such that any torsion in the image of $\phi : M^\vee \to B^\vee(K)$ is annihilated by N. Consider the pullback $\psi : E' \to E$ of $T \to E \to B$ along $[N] : B \to B$ and choose any preimages of basis element of M to define a lattice $M' \subseteq E'$ with $\psi(M') = M$. Then $A := E'/M' \to E/M$ has the property that ϕ' has torsionfree cokernel, hence to has $M_0^\vee \to M^\vee$. □

We deduce:

Lemma 6.3. *We have $\mathcal{O}^\times(E) = K^\times \times M_0^\vee$. Moreover,*

$$\mathrm{Pic}(E) = \mathrm{Pic}(B)/\phi^\vee(M^\vee).$$

Proof. Consider the sheaf $\pi_*\mathcal{O}^\times$ for $\pi : E \to B$. Recall from Lemma 4.3 that for any rigid space U, we have $\mathcal{O}^\times(T \times U) = \mathcal{O}^\times(U) \times M^\vee$. It follows that we have a short exact sequence on B

$$0 \to \mathcal{O}^\times \to \pi_*\mathcal{O}^\times \to \underline{M^\vee} \to 0.$$

Comparing to the above computation of $\mathrm{Hom}(-, \mathbb{G}_m)$, we see that the long exact sequence of cohomology is of the form

$$0 \to K^\times \to \pi_*\mathcal{O}^\times(B) \to M^\vee \xrightarrow{\phi^\vee} B^\vee(K) \to ...$$

Since $\pi_*\mathcal{O}^\times(B) = \mathcal{O}^\times(E)$, this gives the desired description.

The second part follows from the same short exact sequence by [16, Corollary B.8]. □

At this point, we can explicitly describe the Cartan–Leray sequence for the cover $E \to A$:

Theorem 6.4 (rigid Appell–Humbert, [26, Proposition 6.1.9, Example 6.2.1, Theorem 6.3.2])**.** *Let A be an abeloid variety. Then there is a natural short exact sequence*

$$0 \to H^1(M, \mathcal{O}^\times(E)) \to \mathrm{Pic}(A) \to (\mathrm{Pic}(B)/M^\vee)^M \xrightarrow{\delta_2} H^2(M, K^\times).$$

The third term parametrises elements of $\mathrm{Pic}(E)$ *arising from some* $N \in \mathrm{Pic}(B)$ *for which there is a homomorphism* $\lambda : M \to M^\vee$ *such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi^\vee} \\ B & \xrightarrow{\varphi_N} & B^\vee. \end{array} \tag{10}$$

Equivalently, this commutativity means that φ_N *lifts to a map*

$$\varphi_{N,\lambda} : E \to E^\vee.$$

We note that the fourth term in the sequence would a priori be $H^2(M, \mathcal{O}^\times(E))$. In order to understand the boundary map δ_2, and why it factors through $H^2(M, K^\times)$, we to recall a canonical bilinear pairing on $M \times M^\vee$ associated to A:

Let $m \in M^\vee$, this can be interpreted as a morphism $T \to \mathbb{G}_m$. Pushout of E along this map defines a line bundle that is canonically identified with $\mathcal{P}_{m^\vee \times B} \to B$, where $\mathcal{P} \to B \times B^\vee$ is the Poincaré bundle considered as a $\mathbb{G}_m$-torsor. We thus obtain a linear map

$$e_{m^\vee} : E \to \mathcal{P}_{m^\vee \times B} \to B.$$

By swapping the roles of M and $M^\vee$, we see that this is also bilinear in $M^\vee$, in the sense that the above maps for varying $m^\vee \in M^\vee$ assemble to a bilinear map

$$\langle -, - \rangle : M \times M^\vee \to \mathcal{P}_{B \times B^\vee}, \quad (m, m^\vee) \mapsto e_{m^\vee}(m).$$

This can be regarded as a non-vanishing section of $\mathcal{P}_{M \times M^\vee} \to M \times M^\vee$, and in particular gives rise to a canonical trivialisation of $\mathcal{P}$ as a $\mathbb{G}_m$-torsor over $M \times M^\vee \subseteq B \times B^\vee$.

$$\mathcal{P}_{M \times M^\vee} = M \times M^\vee \times \mathbb{G}_m. \tag{11}$$

In particular, this endows the left hand side with the structure of an abelian group (rather than just a biextension).

Suppose now that we are given a pair

$$(N, \lambda) \in \mathrm{Pic}(B) \times \mathrm{Hom}(M, M^\vee)$$

such that the following diagram commutes:

$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ \downarrow & & \downarrow \\ B(K) & \xrightarrow{\varphi_N} & \overline{B}^\vee(K). \end{array} \tag{12}$$

Then the fact that φ_N is symmetric gives rise to a natural isomorphism

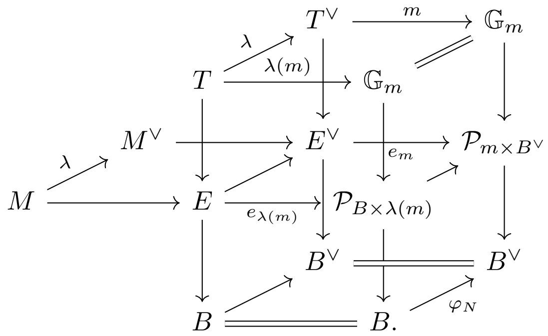
$$\varphi : \varphi_N^* \mathcal{P}_{m, B^\vee} = \mathcal{P}_{B, \lambda(m)}.$$

We can use this to define an alternating pairing

$$\psi_N : M \times M \to K^\times$$

as follows: Fix m and consider the diagram

(13)



We can now use this to define $\psi_N(m, m')$ by sending m' around different corners of the above diagram: We can map

$$M \to E \xrightarrow{e_{\lambda(m)}} \mathcal{P}_{B\times\lambda(m)} \to \mathcal{P}_{m\times B^\vee}$$

and evaluate this at m', this is precisely $\varphi(\langle m', \lambda(m)\rangle)$.

Or we could go via $\lambda : M \to M^\vee$ and use the map $M^\vee \to E^\vee \xrightarrow{e_m} \mathcal{P}_{m\times B^\vee}$. This is precisely $\langle m, \lambda(m')\rangle$.

Now both $\varphi(\langle m', \lambda(m)\rangle)$ and $\langle m, \lambda(m')\rangle$ lie in the fibre $\mathcal{P}_{m\times\lambda(m')}$ of $\mathcal{P} \to B \times B^\vee$ over $(m, \lambda(m'))$. Since this fibre is a homogeneous space under $\mathbb{G}_m(K) = K^\times$, it makes sense to consider their difference (written multiplicatively)

$$\langle m, \lambda(m')\rangle \Big/ \varphi\big(\langle m', \lambda(m)\rangle\big) \in K^\times.$$

This defines the desired alternating pairing

$$\psi_N : M \times M \to K^\times.$$

Proposition 6.5 ([26, Proposition 6.1.9, Example 6.2.1, Theorem 6.3.2])**.** *The connecting map δ_2 in Theorem 6.4 is defined by sending $N \in \mathrm{Pic}(B)$ to the 2-cochain*

$$c_r(N) : M^2 \to K^\times, \quad m_1, m_2 \mapsto r(m_1 + m_2)r(m_1)^{-1}r(m_2)^{-1}r(0)\langle m_1, \lambda(m_2)\rangle$$

where $r : M \to N$ is any trivialisation of ϕ^*N, and the right hand side is considered as an element of $K^\times$ via the canonical identification

$$(\mathrm{id}, \varphi_N)^*\mathcal{P}_{B\times B^\vee} \xrightarrow{\sim} \mathcal{D}_2(N) := m^*N \otimes \pi_1^*N^{-1} \otimes \pi_2^*N^{-1} \otimes 0^*N.$$

Proposition 6.6 ([26, Proposition 6.1.12])**.** *Let (N, λ) be as in Theorem 6.4, i.e. $N \in \mathrm{Pic}(B)$ and $\lambda : M \to M^\vee$ are such that* (10) *commutes. Then the following are equivalent:*

(1) $\delta_2(N) = c_r(N) = 1$,
(2) $\psi_N = 1$,
(3) The following diagram commutes:

(14)
$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow & & \downarrow \\ E & \xrightarrow{\varphi_{N,\lambda}} & E^\vee. \end{array}$$

We can now essentially describe line bundles on A:

Theorem 6.7 (rigid Appell–Humbert, [26, Theorem 6.3.2])**.** *Let A be an abeloid variety. Then there is a natural surjective map between sets of isomorphism classes*

(15)
$$\left\{ \begin{array}{l|l} \textit{triples } (N, \lambda, r) \textit{ consisting of} & \\ N \in \mathrm{Pic}(B), & (10) \textit{ commutes} \\ \lambda : M \to M^\vee \textit{ homomorphism}, & \textit{and } c_r(N) = 1 \\ r : M \to N \textit{ trivialisation} & \end{array} \right\} \to \mathrm{Pic}(A)$$

Given (N,λ)*, a trivialisation* r *exists if and only if* (14) *commutes. The datum of* r *with* $c_r(N)=1$ *is then equivalent to an isomorphism* $r:\phi(m_i)^*N \xrightarrow{\sim} K$ *for each* $i=1,\dots,m$.

Remark 6.8. (1) In terms of this description, we can interpret the exact sequence

$$0 \to H^1(M,\mathcal{O}^\times(E)) \to \mathrm{Pic}(A) \to (\mathrm{Pic}(B)/M^\vee)^M \xrightarrow{\delta_2} H^2(M,K^\times)$$

of Theorem 6.4 as follows: The second map sends (N,λ,r) to N. The condition that this lands in $(\mathrm{Pic}(B)/M^\vee)^M$ remembers the composition $\lambda: M \to M^\vee \to M^\vee/M_0^\vee$. Hence this forgetful map is a torsor under pairs $M \to M_0^\vee$ and homomorphisms $M \to K^\times$ rescaling the trivialisation r. Together, these combine to the datum of a 1-cocycle $M \to \mathcal{O}^\times(E)$, which explains conceptually the kernel $H^1(M,\mathcal{O}^\times(E))$.

(2) The failure of (15) to be a bijection is essentially due to the difference between $\mathrm{Pic}(B)$ and $\mathrm{Pic}(E)=\mathrm{Pic}(B)/M^\vee$. This can be accounted for by equipping A on the right hand side with a certain additional datum of a "cubical trivialisation", see [26, Theorem 6.3.2.] for details. We will not need this in the following.

(3) We think of the condition that $\psi_N=1$ as making precise the idea that $\langle -,\lambda-\rangle$ is symmetric. This is the analogue of the symmetry condition on the Hermitian form in the complex Appell–Humbert Theorem.

By forming quotients in the last diagram of the Appell–Humbert Theorem, we can deduce an explicit description of the Néron–Severi group:

Theorem 6.9 ([26, Theorem 6.3.2])**.** *Let* A *be an abeloid variety. Then*

$$\mathrm{NS}(A) = \mathrm{Hom}(A,A^\vee)^{\mathrm{sym}}$$

is a finite free $\mathbb{Z}$*-module that can be described explicitly as follows: It is the subgroup*

$$\mathrm{NS}(A) \subseteq \mathrm{NS}(B) \times \mathrm{Hom}(M,M^\vee)$$

of pairs (N,λ) *satisfying both of the following two conditions:*

(1) The following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi^\vee} \\ B(K) & \xrightarrow{\varphi_N} & B^\vee(K). \end{array}$$

(2) We have $\psi_N=1$*, i.e.* $\langle m_1,\lambda(m_2)\rangle = \xi\langle m_2,\lambda(m_1)\rangle$.

6.3. **The groups** $\mathcal{O}^\times(E)$ **and** $\overline{\mathcal{O}}^\times(E)$**.** As a preparation for the computation of $\mathrm{Pic}(\widetilde{A})$, we discuss line bundles on perfectoid Raynaud extensions where T is a torus and B is an abeloid variety of good reduction. This essentially means upgrading the results from the last section to the relative setting of relative tori over B. However, as we have seen that $\mathrm{Pic}(\widetilde{T})$ is already huge, we cannot expect $\mathrm{Pic}(\widetilde{E})$ to have a nice description. Instead, we will pass to $\overline{\mathcal{O}}^\times$-torsors, which will be enough since in the compact setting, we can freely switch back and forth between $\mathrm{Pic}(\widetilde{A})$ and $H^1(\widetilde{A},\mathcal{O}^\times[\frac{1}{p}])$.

We start by noting a phenomenon that will be relevant when we compare line bundles on $\widetilde{A}$ to those on A: While for the torus T, we have $\mathcal{O}(T)^\times[\frac{1}{p}] = \overline{\mathcal{O}}^\times(T)[\frac{1}{p}]$, the group $\overline{\mathcal{O}}^\times(E)$ will in general be larger than $\mathcal{O}^\times(E)[\frac{1}{p}]$.

Definition 6.10. Recall that E defines an extension class in $\mathrm{Ext}^1_{\mathrm{an}}(B,T)$ corresponding to a morphism

$$M^\vee \to B^\vee(K).$$

Let $M_0^\vee$ be the kernel of this morphism, this is a finite free $\mathbb{Z}$-module of rank in between 0 and r. The injection $M_0^\vee \to M^\vee$ corresponds to a morphism of tori $T \to T_0$ which is universal with the property that the pushout $T_0 \times^T E$ is split.

Definition 6.11. By Theorem 5.10, $\widetilde{E}$ defines a class in $\mathrm{Ext}^1_{\mathrm{an}}(\widetilde{B},\widetilde{T})$ which by Corollary 5.5 corresponds to a homomorphism

$$M^\vee[\tfrac{1}{p}] \to B^\vee(k)[\tfrac{1}{p}].$$

Let $M_1^\vee$ be its kernel. This is a finite free $\mathbb{Z}[\frac{1}{p}]$-module of rank in between 0 and r.

The two modules $M_0^\vee$ and $M_1^\vee$ fit into a commutative diagram

$$\begin{array}{ccccc} M_0^\vee & \hookrightarrow & M^\vee & \longrightarrow & B^\vee(K) \\ \downarrow & & \downarrow & & \downarrow \\ M_1^\vee & \hookrightarrow & M^\vee[\frac{1}{p}] & \longrightarrow & B^\vee(k)[\frac{1}{p}]. \end{array}$$

In particular, we have an inclusion

$$M_0^\vee[\tfrac{1}{p}] \subseteq M_1^\vee,$$

which in general may be proper: This happens e.g. if $\widetilde{E}$ becomes split, while E is not.

The injection $M_1^\vee \to M^\vee[\frac{1}{p}]$ corresponds to a morphism of tori $\widetilde{T} \to \widetilde{T}_1$ where $\widetilde{T}_1$ is the perfectoid torus with character lattice $M_1^\vee$. This morphism is universal with the property that the pushout $\widetilde{T}_1 \times^{\widetilde{T}} \widetilde{E}$ is split.

Definition 6.12. We set

$$M_2^\vee := M^\vee[\tfrac{1}{p}]/M_1^\vee.$$

Equivalently, this can be described as the image of $M^\vee[\frac{1}{p}]$ in $\overline{B}(k)[\frac{1}{p}] \subseteq \mathrm{Pic}(\widetilde{B})$.

6.4. **$\overline{\mathcal{O}}^\times$-torsors on E.** We now pass from $\mathcal{O}^\times$ to $\overline{\mathcal{O}}^\times$ since the latter is easier to control for relative tori.

Lemma 6.13. *Let Z be any rigid space and let $\pi : E \to Z$ be a T-torsor. Then we have short exact sequences of sheaves on Z*

$$0 \to \overline{\mathcal{O}}^\times \to \pi_*\overline{\mathcal{O}}^\times \to M^\vee[\tfrac{1}{p}] \to 0.$$

This sequences is not in general split unless π is a trivial torsor.

Proof. Let $U \to Z$ be any affinoid open over which π is split. By Lemma 4.3,

$$\overline{\mathcal{O}}^\times(T \times U) = \overline{\mathcal{O}}^\times(U) \times M^\vee[\tfrac{1}{p}],$$

which glues to give the desired exact sequence. ☐

The main result of this section is now the following description of $\overline{\mathcal{O}}^\times$-torsors on E:

Proposition 6.14. *We have a short exact sequence*

$$0 \to K^\times/(1+\mathfrak{m}) \to H^0(E, \overline{\mathcal{O}}^\times) \to M_1^\vee \to 0$$

as well as an isomorphism

$$H^1(E, \overline{\mathcal{O}}^\times) = \mathrm{Pic}(\overline{B})/M_2^\vee.$$

Proof. Since $E \to B$ is a T-torsor, we have

$$H^1(E, \overline{\mathcal{O}}^\times) = H^1(B, \pi_*\overline{\mathcal{O}}^\times)$$

by Proposition 4.5. By Lemma 6.13, this fits into an exact sequence of étale cohomology

$$\overline{\mathcal{O}}^\times(E) \to M^\vee[\tfrac{1}{p}](X) \to H^1(B, \overline{\mathcal{O}}^\times) \to H^1(B, \pi_*\overline{\mathcal{O}}^\times) \to H^1(B, M^\vee[\tfrac{1}{p}]).$$

The last term vanishes since $H^1(B, \mathbb{Z}) = 0$ by [6, Theorem 1.2.(c)].

The image of the second map is $M^\vee[\frac{1}{p}]/M_1^\vee = M_2^\vee$ by definition of $M_1^\vee$ and $M_2^\vee$ in §6.3. ☐

6.5. **Universal covers of abeloid varieties.** In this subsection, let K be an algebraically closed perfectoid field of characteristic 0 or p.

Let A be a connected smooth proper rigid group variety over K, always assumed to be commutative. Such objects are called abeloid varieties. The algebraic (equivalently, projective) abeloid varieties are precisely the abelian varieties.

Let us recall the main statements about uniformisation of abeloid varieties, due to Raynaud and Bosch–Lütkebohmert: By the theory of Raynaud extensions, one can associate to A a connected rigid group variety $E \to A$ that is universal with the property that $H^1_{\mathrm{an}}(E, \mathbb{Z}) = 0$. It fits into a short exact sequence with respect to the analytic topology

$$0 \to T \to E \to B \to 0 \tag{16}$$

of analytic adic spaces over K, where B is an abelian variety with good reduction and T is a rigid torus of rank r (necessarily split since K is algebraically closed). Moreover, there is a lattice $\underline{\mathbb{Z}}^r \cong M \subseteq E$ such that there is a short exact sequence in the analytic topology,

$$0 \to M \to E \to A \to 0. \tag{17}$$

The data of these two exact sequences is commonly referred to as the Raynaud uniformisation of A. Translated into the setting of diamonds, we may equivalently regard (16) and (17) as exact sequences of abelian sheaves on Perf_K.

In this setting, in analogy to the rigid analytic universal cover, there is a p-adic perfectoid universal cover of A: By the Main Theorem of [3], this has a perfectoid tilde-limit

$$\widetilde{A} = \varprojlim_{[p]} A.$$

We also recall the following result, which explains in what sense $\widetilde{A}$ is the universal cover:

Proposition 6.15 ([16, Theorem 3.10, Corollary 3.11])**.** *For any abeloid variety A,*

$$H^i_v(\widetilde{A}, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{for } i = 0, \\ 0 & \text{for } i > 0, \end{cases}$$

$$H^i_v(\widetilde{A}, \mathcal{O}^+) \stackrel{a}{=} \begin{cases} \mathcal{O}_K & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

In particular, we have the following link between the theory in characteristic 0 and p:

Corollary 6.16. *The sharp map $\sharp : \mathrm{Pic}(\widetilde{A}^\flat) \to \mathrm{Pic}(\widetilde{A})$ is an isomorphism.*

Proof. This follows from the short exact sequence on $\widetilde{A}_v$

$$0 \to \mathbb{Z}_p(1) \to \mathcal{O}^{\flat\times} \to \mathcal{O}^\times \to 0$$

using Proposition 6.15. □

Corollary 6.17. *We have* $\mathrm{Pic}(\widetilde{A}) = H^1(\widetilde{A}, \overline{\mathcal{O}}^\times)$.

Proof. It suffices to see that $H^i(X, 1 + \mathfrak{m}\mathcal{O}^+) = 0$ for $i = 1, 2$. In characteristic 0, this follows from Proposition 6.15 by the logarithm sequence

$$0 \to \mu_p^\infty \to 1 + \mathfrak{m}\mathcal{O}^+ \to \mathcal{O} \to 0.$$

In characteristic p, it follows from the analogous sequence on $\widetilde{A}_v$

$$0 \to \mathbb{Q}_p \to 1 + \mathfrak{m}\mathcal{O}^+ \to \mathcal{O}^\sharp \to 0.$$

□

In particular, this shows that every line bundle on $\widetilde{A}$ can be lifted to a $\widetilde{\mathbb{G}}_m$-torsor in a unique way, i.e. every line bundle has a natural perfectoid cover.

Building on the previous sections, we now determine the group $\mathrm{Pic}(\widetilde{A})$ in general. Our strategy is inspired by the complex case, where for any complex torus the short exact sequence of the universal cover

$$0 \to \Lambda \to \mathbb{C}^g \to \mathbb{C}^g/\Lambda \to 0$$

gives rise to a description of the Picard group as group cohomology

$$\mathrm{Pic}(\mathbb{C}^g/\Lambda) = H^1(\Lambda, \mathcal{O}^\times(\mathbb{C})).$$

This cohomology group can be described explicitly using the complex exponential, leading to what is known as the Appell–Humbert Theorem.

There is an analogous description for abeloid varieties, using instead the sequence

$$0 \to M \to E \to A,$$

as we shall recall in Theorem 6.4 below.

In order to pass to the perfectoid case, it is possible in principle to carry out a similar strategy with the short exact sequence

$$0 \to M \to M_p \times \widetilde{E} \to \widetilde{A} \to 0.$$

However, the terms arising in the Appell–Humbert sequence are more difficult to control: While $\mathrm{Pic}(E)$ is manageable, we have seen that there are many line bundles on $\widetilde{E}$. A second issue is that the group cohomology is of the form

$$H^1_{\mathrm{cts}}(M, \mathcal{C}(M \otimes \mathbb{Z}_p, K^\times))$$

where M acts as translation. It it possible to show that this equals

$$\mathrm{Hom}(M, K^\times/(1+\mathfrak{m})),$$

but the proof is a bit involved.

Instead, we shall therefore reduce attention to $\overline{\mathcal{O}}^\times$-torsors from the beginning, which also allows us to work on A instead of $\widetilde{A}$. This allows for a treatment which is closer to the rigid situation.

For the entirety of this section, let us fix an abeloid A over K with dual $A^\vee$ and Raynaud uniformisations

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ T & \longrightarrow & E & \longrightarrow & B, \\ & & \downarrow & & \\ & & A & & \end{array} \qquad \begin{array}{ccccc} & & M^\vee & & \\ & & \downarrow & & \\ T^\vee & \longrightarrow & E^\vee & \longrightarrow & B^\vee, \\ & & \downarrow & & \\ & & A^\vee & & \end{array}$$

as in §3.1. Recall from [26, Theorem 7.6.4], [25, §9] that the extension classes $E \in \mathrm{Ext}^1(B, T)$ and $E^\vee \in \mathrm{Ext}^1(B^\vee, T^\vee)$ correspond to morphisms

$$\phi^\vee : M^\vee \to B^\vee(K), \quad \phi : M \to B(K),$$

respectively. For a second abeloid A', we are going to use analogous notation, i.e. $E', M', \dots$.

6.6. **Lifting $\overline{\mathcal{O}}^\times$-torsors to line bundles.** As we have just seen, in the rigid setting, any line bundle N on B that defines an element in $(\mathrm{Pic}(B)/M)^M$ gives rise to an alternating bilinear pairing

$$\psi_N : M \times M^\vee \to K^\times$$

which can be interpreted as a 2-cocycle in group cohomology.

For the perfectoid Appell–Humbert Theorem, we claim that we instead have for any $N \in \mathrm{Pic}(\widetilde{B}_p) = (\mathrm{Pic}(\widetilde{B}_p)[\frac{1}{p}]/M_1^\vee)^M$ an alternating bilinear pairing

$$\overline{\psi}_N : M \times M^\vee \to K^\times/(1+\mathfrak{m}).$$

Proposition 6.18. *Let N be a line bundle on $\widetilde{B}_p$ and $\lambda : M \to M^\vee$ such that the following diagram commutes:*

$$\begin{array}{ccc} M[\tfrac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\tfrac{1}{p}] \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \overline{\phi}^\vee} \\ \overline{B}(k)[\tfrac{1}{p}] & \xrightarrow{\varphi_N} & \overline{B}^\vee(k)[\tfrac{1}{p}]. \end{array}$$

Then there is an abeloid variety A' with uniformisation $A' = E'/M'$, $T' \to E' \to B'$ and with an isomorphism $\widetilde{A}'_p \xrightarrow{\sim} \widetilde{A}_p$ inducing an isomorphism $\widetilde{B}'_p \xrightarrow{\sim} \widetilde{B}_p$, with respect to which N is the pullback of a line bundle N' on B' such that the following lift of this diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi'^\vee} \\ B'(K) & \xrightarrow{\varphi_{N'}} & B'^\vee(K)[\tfrac{1}{p}]. \end{array} \tag{18}$$

We can always do this in such a way that $\ker \phi'^\vee[\frac{1}{p}] = \ker \overline{\phi}^\vee$.

Proof. By Corollary 6.20 below, we can find a lift B' of $\overline{B}$ such that N comes via pullback from a line bundle N' on B'.

We now choose compatible lifts of $\overline{\phi}$ and $\overline{\phi^\vee}$ as follows. For later applications, let us axiomatize the argument:

Lemma 6.19. *Assume that we are in one of the following situations:*

1. *(1) We have for each abeloid variety A a subgroup $U(A) \subseteq A(K)$ such that any homomorphism $A \to A'$ sends $U(A) \to U(A')$, for example $U(A) = \widehat{A}$. We write $\overline{A}(K) := A(K)/U(A)$. Or:*
2. *(2) We are given subgroups $U(B) \subseteq B(K)$ and $U(B^\vee) \subseteq B(K)$ such that with notation as above, φ_N and $\overline{\varphi}_N : \overline{B}(K) \to \overline{B}^\vee(K)$ are both injective.*

Assume that we have a line bundle N on B and a morphism $\lambda : M \to M^\vee$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \overline{\phi}^\vee} \\ \overline{B}(K) & \xrightarrow{\overline{\varphi}_N} & \overline{B}^\vee(K). \end{array}$$

Then we can find $\phi' : M \to B(K)$ and $\phi'^\vee : M^\vee \to B^\vee(K)$ such that all of the following hold up to torsion: $\phi' \equiv \phi \mod U(B)$ and $\phi'^\vee \equiv \phi^\vee \mod U(B^\vee)$ and $\ker \phi'^\vee = \ker \overline{\phi}^\vee$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi'^\vee} \\ B(K) & \xrightarrow{\varphi_N} & B^\vee(K). \end{array}$$

Proof. We first observe that we may without loss of generality make $U(A)$ bigger and replace it by the kernel of $A(K) \to A(K)/U(A) \otimes \mathbb{Q}$. Then $\overline{A} = A(K)/U(A)$ is torsionfree, but also divisible since $A(K)$ is. Hence it is a $\mathbb{Q}$-vector space. Consequently, the given diagram extends uniquely to

$$\begin{array}{ccc} M \otimes \mathbb{Q} & \xrightarrow{\lambda\otimes\mathbb{Q}} & M^\vee \otimes \mathbb{Q} \\ \downarrow{\scriptstyle \overline{\phi}\otimes\mathbb{Q}} & & \downarrow{\scriptstyle \overline{\phi}^\vee\otimes\mathbb{Q}} \\ \overline{B}(K) & \xrightarrow{\overline{\varphi}_N} & \overline{B}^\vee(K). \end{array}$$

Let $B_0 := \ker \varphi_N$ and $B_1 := \operatorname{im} \varphi_N \subseteq B^\vee$. Both of these are abeloid varieties, hence we obtain an extension

$$0 \to B_0 \to B \to B_1 \to 0.$$

Any such extension is split on a finite cover, hence we can find an isogeny

$$B_0 \times B_1 \to B.$$

Similarly, we can find an isogeny

$$B^\vee \to B_1 \times B_2$$

such that φ_N is up to isogenies given by the identity $\mathrm{id} : B_1 \to B_1$.

By functoriality of U and hence of $\overline{\cdot}(K)$, this induces a map

$$\overline{B_0}(K) \times \overline{B_1}(K) \to \overline{B}(K)$$

which maps $\overline{B_1}$ onto its isomorphic image since everything is torsionfree.

Let $M_1^\vee := \ker \overline{\phi}^\vee \otimes \mathbb{Q}$. By our assumption that $\ker \phi'^\vee = \ker \overline{\phi}^\vee$, we need to set $\phi'^\vee|_{M_1^\vee} = 0$.

Let M_1 be the preimage of $M_1^\vee$ under $\lambda \otimes \mathbb{Q}$. Choose any lift of $\overline{\phi'}$ to $M \to \overline{B_0}(K) \times \overline{B_1}(K)$. Then the fact that the diagram commutes means that

$$\tilde{\overline{\phi}}' : M \to \overline{B_0}(K) \times \overline{B_1}(K) \to \overline{B}^\vee(K)$$

sends M_1 to 0. Consequently, M_1 lands in $\overline{B_0} \subseteq \overline{B_0}(K) \times \overline{B_1}(K)$. We can define ϕ' on M_1 to be any lift $\phi' : M_1 \otimes \mathbb{Q} \to B_0(K) \to B(K)$ of $\overline{\phi}$. Then $\varphi_N \circ \phi'(M_1) = 0$ by construction.

By completing from $M_1 \otimes \mathbb{Q}$ to a basis of $M \otimes \mathbb{Q}$, we now extend this to *any* lift

$$\phi' : M \otimes \mathbb{Q} \to B'(K)$$

of $\overline{\phi}$. We claim that we can now find a compatible lift

$$\phi'^\vee : M^\vee \otimes \mathbb{Q} \to B'^\vee(K)$$

making the desired diagram commute. We had already defined $\phi'^\vee$ on $M_1^\vee \otimes \mathbb{Q}$.

On the $\mathbb{Q}$-subvector space given by the image of λ, the map $\overline{\phi}'^\vee$ is uniquely determined by commutativity of the diagram by taking any preimage in $M \otimes \mathbb{Q}$ and sending it around the bottom corner of the diagram. This does not depend on the choice of preimage since $\ker \lambda \otimes \mathbb{Q} \subseteq M_1 = \lambda^{-1}(M_1^\vee)$ is sent to 0 in $B'^\vee(K)$ by construction.

Finally, we simply choose any sub-vector space $L \subseteq M^\vee \otimes \mathbb{Q}$ such that $M \otimes \mathbb{Q} = L \oplus (\operatorname{im} \lambda \otimes \mathbb{Q} + M_1^\vee \otimes \mathbb{Q})$ and choose *any* lift of $\overline{\phi}^\vee|_L : L \to \overline{B}^\vee(K)$ to $\phi'^\vee|_L : L \to B'(K)$. This is necessarily injective since it is on the reduction.

By construction, our different functions on L, $\operatorname{im}(\lambda \otimes \mathbb{Q})$ and $M_1^\vee$ now glue together to a function $\phi'^\vee : M_1^\vee \to B'(K)$. Then since $\operatorname{im}(\lambda) \cap L = 0$, the desired diagram commutes, and the functions satisfy all desired properties. □

Returning to the setup of Proposition 6.18, we can furthermore deal with the torsion in Lemma 6.19: Since replacing M by $p^n M$ and $M^\vee$ by $p^n M^\vee$ for any $n \in \mathbb{N}$ yields an abeloid variety with cover isomorphism to $\widetilde{B}_p$, we may without loss of generality assume that the congruences and commutativity of the Lemma hold up to coprime-to-p torsion. Observe now that the reduction map $B(K) \to \overline{B}(k)[\frac{1}{p}]$ identifies the N-torsion subgroups for any $N \in \mathbb{N}$ coprime to p. Hence we can uniquely lift $\overline{\phi} - \overline{\phi}'$ to a morphism $\delta : M \to B(K)$ and replace ϕ' by $\phi' + \delta$ to ensure that $\phi \equiv \phi' \bmod \widehat{B}(K)$. Similarly for $\phi'^\vee$.

Finally, if the diagram commutes up to coprime-to-p-torsion in $B^\vee(K)$, then the fact that reduction mod $\widehat{B}^\vee(K)$ identifies coprime-to-p torsion implies that it already commutes.

The maps ϕ' and $\phi'^\vee$ of the Lemma now define Raynaud extensions $T^\vee \to E'^\vee \to B'^\vee$ and $T \to E' \to B'$ lifting $\overline{E}^\vee$ and $\overline{E}$. The commutativity of the diagram means that (N, λ) defines an isogeny $E' \to E'^\vee$. Finally, we can lift $M \to B'(K)$ to a map $M \to E'(K)$ in such a way that this agrees with $M \to E(K)$ on the associated $\overline{\mathcal{O}}^\times$-torsor. Then the abeloid

$$A' := E'/M$$

is as desired, and the map $\overline{E} \to \overline{E}'$ defines an isomorphism $A' \xrightarrow{\sim} A$ by [16, Theorem 5.4] since $M \to E$ and $M \to E'$ are identified on the special fibre.

□

The following Lemma was used in the proof:

Lemma 6.20. *For any line bundle L on $\widetilde{B}_p$, there is an abelian variety B' over K together with an isomorphism $\widetilde{B}'_p \xrightarrow{\sim} \widetilde{B}_p$ and a line bundle L' on B' such that L is the pullback of L'.*

Proof. We can without loss of generality assume that L is the image of a line bundle $\overline{L}$ on $\overline{B}$ via the map

$$\mathrm{Pic}(\overline{B}) \subseteq \mathrm{Pic}(\overline{B})[\tfrac{1}{p}] \xrightarrow{\sim} \mathrm{Pic}(\widetilde{B}_p).$$

The general case then follows since we can always extract p-th roots of line bundles via $[p^n]$ by [15, Lemma 4.7].

By a theorem of Mumford [31, Theorem (2.3.3)], there exists a lift $(\mathfrak{B}', L')$ of $(\overline{B}, \overline{L})$ to $\mathcal{O}_K$, i.e. a formal abelian scheme $\mathfrak{B}'$ together with an isomorphism $\phi : \mathfrak{B}'_k \xrightarrow{\sim} \overline{B}$ and a line bundle L' on $\mathfrak{B}'$ whose special fibre is isomorphic to $\phi^*\overline{L}$. Let B' be the generic fibre, then by (the easy direction of) [16, Theorem 4.2], the isomorphism ϕ induces an isomorphism

$$\widetilde{B}'_p \xrightarrow{\sim} \widetilde{B}_p$$

that identifies the pullback of L' with L, as desired. □

Coming back to our pairing, Proposition 6.18 now allows us to assume without loss of generality that N is the pullback of a line bundle N' on $\mathrm{Pic}(B)$. We can then simply define $\overline{\psi}_N$ to be the reduction of $\psi_{N'}$ mod $1+\mathfrak{m}$. This is independent of the choices since any other choice of B' and N' is quasi-isogeneous on the special fibre, and thus gives rise to the same pairing with values in $K^\times/(1+\mathfrak{m})$.

Alternatively, we can give a more intrinsic definition as follows: We first note:

Lemma 6.21. *Let (N, λ) be a pair of $N \in \mathrm{Pic}(\widetilde{B}_p) = \mathrm{Pic}(\overline{B})[\frac{1}{p}]$ and a group homomorphism $\lambda : M[\frac{1}{p}] \to M^\vee[\frac{1}{p}]$. Then N and λ define quasi-isogenies $\varphi_N : \overline{B} \to \overline{B}^\vee$ and $\lambda : \overline{T} \to \overline{T}^\vee$ respectively, and the following are equivalent:*

(1) The following diagram commutes

$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ \downarrow & & \downarrow \\ B(k)[\frac{1}{p}] & \xrightarrow{\varphi_N} & \overline{B}^\vee(k)[\frac{1}{p}]. \end{array} \tag{19}$$

(2) There is an isogeny of $\overline{\mathcal{O}}^\times$-torsors $\overline{E} \to \overline{E}^\vee_k$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{T} & \longrightarrow & \overline{E} & \longrightarrow & \overline{B} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\varphi_{N,\lambda}} & & \downarrow{\scriptstyle\varphi_N} & & \\ 0 & \longrightarrow & \overline{T}^\vee & \longrightarrow & \overline{E}^\vee & \longrightarrow & \overline{B}^\vee & \longrightarrow & 0 \end{array}$$

where $\overline{T} = T/\widehat{T}$ and $\overline{E} = E/\widehat{E}$, and similarly for the duals.

Proof. It suffices to see that we have

$$\mathrm{Ext}^1_{\mathrm{Zar}}(\overline{B}, \overline{\mathcal{O}}^\times) = \overline{B}(k)[\tfrac{1}{p}],$$

which implies $\mathrm{Ext}^1_{\mathrm{Zar}}(\overline{B}, \overline{T}) = \mathrm{Hom}(M[\frac{1}{p}], \overline{B}(k)[\frac{1}{p}])$ since $M[\frac{1}{p}] = \mathrm{Hom}(\overline{T}, \overline{\mathcal{O}}^\times)$ by Lemma 4.3.5.

But the Ext-group are precisely the translation invariant elements of $H^1_{\mathrm{Zar}}(\overline{B}, \overline{\mathcal{O}}^\times)$, which is $= \mathrm{Pic}(\overline{B})[\frac{1}{p}]$ by [15, Proposition 3.10]. □

Let now (N, λ) be a pair satisfying the equivalent conditions of the lemma. Let $\mathcal{P}$ be the pullback of the Poincaré bundle to $B \times B^\vee$ and let $\overline{\mathcal{P}}$ be its reduction to a $\overline{\mathcal{O}}^\times$-torsor on $\overline{B} \times \overline{B}^\vee$. To any $m \in M[\frac{1}{p}]$, we have an associated morphism

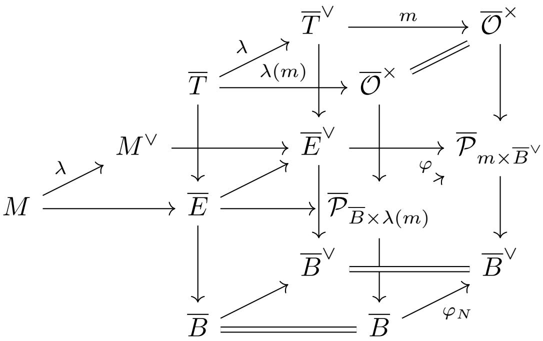
$$m : \overline{T}^\vee \to \overline{\mathcal{O}}^\times.$$

Pushout of $\overline{E}$ along this morphism defines an extension

$$0 \to \overline{\mathcal{O}}^\times \to \overline{\mathcal{P}}_{m \times \overline{B}} \to \overline{B}^\vee \to 0.$$

Consider now the pullback of $\overline{\mathcal{P}}_{m\times\overline{B}}$ along φ_B. Fix $m \in M$, then using the lemma, we see that this fits into a diagram of commutative cubes

(20)



in which all squares except possibly the leftmost one commute. The fact that $\varphi_B^\vee = \varphi_B$ is symmetric implies that we have

$$\overline{\mathcal{P}}_{\overline{B}\times\lambda(m)} \cong m^*N \otimes N^{-1}.$$

We now repeat the construction after (13) for the diagram (20) to get the desired pairing

$$\begin{aligned} \overline{\psi}_{N,\lambda} \colon M \times M \to & \overline{\mathcal{O}}^\times(K) = K^\times/(1+\mathfrak{m}) \\ m, m' \mapsto & \langle m, \lambda(m')\rangle \Big/ \varphi\big(\langle m', \lambda(m)\rangle\big). \end{aligned}$$

It is clear from comparing the two diagrams (13) and (20) that for any lift N' as above, the reduction of $\psi_{N'}$ mod $1+\mathfrak{m}$ agrees with $\overline{\psi}_N$.

Lemma 6.22. *Let (N,λ) be such that the conditions of Lemma 6.21 are satisfied. In particular, we have a quasi-isogeny $\varphi_{N,\lambda} : \overline{E} \to \overline{E}^\vee$. Associated to this we have by* [16, Theorem 5.2] *an isomorphism $\widetilde{\varphi}_{N,\lambda} : \widetilde{E}_p \to \widetilde{E}_p^\vee$. Then the following are equivalent:*

(1) We have $\overline{\psi}_{N,\lambda} = 1$, that is $\langle m, \lambda(m')\rangle = \varphi\big(\langle m', \lambda(m)\rangle\big)$ for all m, m'.
(2) The following diagram commutes:

(21)
$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \overline{E}(k) & \xrightarrow{\varphi_{N,\lambda}} & \overline{E}^\vee(k). \end{array}$$

(3) The following diagram commutes up to topological p-torsion

(22)
$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \widetilde{E}_p(K) & \xrightarrow{\widetilde{\varphi}_{N,\lambda}} & \widetilde{E}_p^\vee(K). \end{array}$$

Proof. The $\overline{T}^\vee$-torsor $\overline{E}^\vee$ can be written as the iterated fibre product over $\overline{B}^\vee$ of $\mathcal{P}_{m_i\times B^\vee}$ where $m_1,\dots,m_r$ is a basis of M. Consequently, it suffices to check that the outer horizontal diagram in the middle row of (20) commutes for all m. This happens precisely if $\psi_N=1$ by definition of the latter. This shows the equivalence of 1 and 2.

The equivalence of 2 and 3 follows since the topological p-torsion elements of $\widetilde{E}^\vee_p(K)$ are precisely those in the kernel of $\widetilde{E}^\vee_p(K)\to E(K)\to\overline{E}(k)[\frac{1}{p}]$. □

Proposition 6.23. *In the setting of Proposition 6.18, let $L\subseteq\mathcal{O}_K^\times$ be some divisible subgroup and let $\overline{T}(K)=\mathrm{Hom}(M^\vee,K^\times/L)$. Suppose that we have a quotient map $E(K)\to\overline{E}(K)$ such that $\overline{E}(K)$ is an extension*

$$0\to\overline{T}(K)\to\overline{E}(K)\to\overline{B}(K)\to 0,$$

and similarly for $E^\vee\to\overline{E}^\vee$ with $\overline{T}^\vee(K)=\mathrm{Hom}(M,K^\times/L)$. Assume that the morphism $\varphi:E\to E^\vee$ reduces to a morphism $\overline{\varphi}:\overline{E}\to\overline{E}^\vee$ and that we have morphisms $\overline{h}:M\to\overline{E}(K)$ and $\overline{h}^\vee:M^\vee\to\overline{E}^\vee(K)$ making the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow & & \downarrow \\ \overline{E}(K) & \xrightarrow{\varphi_N} & \overline{E}^\vee(K). \end{array} \tag{23}$$

Then one can find lifts $h:M\to E$ of h and $h^\vee:M^\vee\to E^\vee$ of $h^\vee$ which making the following diagram commute

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow & & \downarrow \\ E(K) & \xrightarrow{\varphi_{N'}} & E^\vee(K) \end{array} \tag{24}$$

and such that the resulting abeloid E/M is dual to $E^\vee/M^\vee$.

Proof. By Lemma 6.22, the square (23) that we wish to lift commutes if and only if $\overline{\psi}_{N,\lambda}=1$. Given any lifts h and $h^\vee$, the square (24) commutes if and only if $1=\psi_{N',\lambda}$, where

$$\psi_{N',\lambda}:\wedge^2M\to K^\times,\quad (m_1,m_2)\mapsto\varphi_{N'}(\langle m_1,\lambda(m_2)\rangle)\Big/\langle m_2,\lambda(m_1)\rangle. \tag{25}$$

Let us denote the kernels of $E(K)\to\overline{E}(K)$ and $E(T)\to\overline{T}(K)$ etc by $\widehat{E}(K)$, $\widehat{E}(K)$, etc. Then by assumption, the diagram (24) commutes up to a difference of the form

$$M\to\widehat{E}(K).$$

We wish to perturb our lift $h:M\to E$ in such a way that this difference is cancelled out, so that (24) commutes. But in doing so, we need to be careful that changing $M\to E'$ also changes the map $M^\vee\to E'^\vee$ in (24). In order to control both $M\to E'$ and $M^\vee\to E'^\vee$ simultaneously, we switch perspective using [26, Proposition 6.1.8] and consider instead the equivalent datum of the bilinear pairing

$$\langle -,-\rangle_{A'}:M\times M^\vee\to\mathcal{P}_{B'\times B'^\vee}$$

determined by (and conversely, determining) $A':=E/M$. As the composition $\phi'\times\phi'^\vee$: $M\times M^\vee\to B\times B^\vee$ is fixed by our choices of E' and $E'^\vee$, the remaining ambiguity in A' is a scaling in each of the fibres of $\mathcal{P}_{B'\times B'^\vee}\to B\times B^\vee$. More precisely, we can perturb $\langle -,-\rangle_{A'}$ by multiplying with any bilinear map

$$\delta:M\times M^\vee\to L$$

and obtain a pairing

$$\langle -,-\rangle_\delta:=\langle -,-\rangle\cdot\delta:M\times M^\vee\to\mathcal{P}_{B'\times B'^\vee}$$

defining an abeloid that is "p-adically close to A'". Explicitly, we can regard δ as a morphism $\delta:M\to\widehat{T}(K)$, and the abeloid corresponding to $\langle -,-\rangle_{A',\delta}$ is the one defined by the lattice

$M \to E'$ obtained by translating the lattice of A' by the topologically p-torsion difference $M \xrightarrow{\delta} \widehat{T}(K) \to \widehat{E}(K)$.

Observe now that for the resulting pairing $\langle -, - \rangle_\delta$, the pairing $\psi_{N,\lambda}$ becomes

$$\psi_{N',\lambda,\delta} : \wedge^2 M \to K^\times, \quad m_1, m_2 \mapsto \langle m_1, \lambda(m_2)\rangle \cdot \delta(m_1, \lambda(m_2)) \Big/ \varphi\big(\langle m_2, \lambda(m_1)\rangle \cdot \delta(m_1, \lambda(m_2))\big)$$

$$= \psi_{N,\lambda} \cdot \Big(\delta(m_1, \lambda(m_2))/\delta(m_2, \lambda(m_1))\Big)$$

where we have used that φ is $K^\times$-linear. It thus suffices to prove that there is a morphism

$$\delta : M \times M^\vee \to L$$

such that

$$\delta(m_1, \lambda(m_2))/\delta(m_2, \lambda(m_1)) = \psi_{N,\lambda}^{-1},$$

as then $\langle -, - \rangle_\delta$ will define a pairing for which $\psi_{N',\lambda,\delta} = 1$, as desired.

At this point, we have reduced to a linear algebra problem: Since $\psi_{N,\lambda}^{-1}$ is alternating, Lemma A.2 shows that it is always possible to find a bilinear map

$$\epsilon : M \times M \to L$$

such that $\psi_{N,\lambda}^{-1}(m_1 \wedge m_2) = \epsilon(m_1, m_2) - \epsilon(m_2, m_1)$. Given such an ϵ, we are left to see that one can find $\lambda : M \times M^\vee \to 1 + \mathfrak{m}$ such that $\epsilon(-,-) = \delta(-, \lambda -)$.

Since $\mathrm{Hom}(M, L)$ is divisible, this is possible if and only if ϵ factors through $M \times M/N$ where $N := \ker \lambda$. Note that since $M^\vee$ is torsionfree, $N \subseteq M$ is a split submodule. By Lemma A.4, it now suffices to see that $\psi_{N,\lambda}^{-1}$ vanishes on $\wedge^2 N$, which is clear from (25). □

6.7. **The perfectoid Appell–Humbert Theorem.** We are now ready to prove the analogue of the Appell–Humbert Theorem in our setting: We start with the intermediate cover

$$\widetilde{A}_p := \varprojlim_{[p]} A.$$

Theorem 6.24 (Appell–Humbert for $\widetilde{A}_p$)**.** *The Cartan–Leray sequence of $\overline{\mathcal{O}}^\times$ applied to the analytic M-torsor $E \to A$ induces an exact sequence*

$$0 \to H^1(M, \overline{\mathcal{O}}^\times(E)) \to \mathrm{Pic}(\widetilde{A}_p) \to (\mathrm{Pic}(\widetilde{B})/M^\vee[\tfrac{1}{p}])^M \xrightarrow{\delta_2} \mathrm{Hom}(\wedge^2 M, K^\times/(1+\mathfrak{m})).$$

where δ_2 is defined as in Proposition 6.5.

Alternatively, we may regard this sequence as the Cartan–Leray sequence for the sheaf $\mathcal{O}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ applied to the analytic M-torsor $X \to \widetilde{A}$.

Theorem 6.25. *For any $N \in \mathrm{Pic}(\widetilde{A})$, there is an abeloid A' with an isomorphism $\widetilde{A}' \xrightarrow{\sim} \widetilde{A}$ such that N is in the image of* $\mathrm{Pic}(A') \to \mathrm{Pic}(\widetilde{A})$.

Corollary 6.26. *Every line bundle on $\widetilde{B}$ has a formal model on $\widetilde{\mathfrak{B}}$.*

Corollary 6.27. *Any line bundle on $\widetilde{A}$ is integral, i.e. comes from an $\mathcal{O}^{\times,+}$-torsor.*

While Corollary 6.25 follows from Theorem 6.24 by comparing linear algebra data, our proves of these two statements will go hand in hand:

Proof of Theorem 6.24 and Theorem 6.25. By Corollary 6.17, we have

$$H^1(\widetilde{A}, \mathcal{O}^\times) = H^1(\widetilde{A}, \overline{\mathcal{O}}^\times).$$

By [14, Corollary 3.20], we have

$$H^1(\widetilde{A}, \overline{\mathcal{O}}^\times) = \varinjlim_{[N]} H^1(A, \overline{\mathcal{O}}^\times).$$

The group $H^1(A, \overline{\mathcal{O}}^\times)$ sits in a Cartan–Leray spectral sequence for $E \to A$:

$$0 \to H^1(M, \overline{\mathcal{O}}^\times(E)) \to H^1(A, \overline{\mathcal{O}}^\times) \to H^1(E, \overline{\mathcal{O}}^\times)^M \xrightarrow{\delta_2} H^2(M, \overline{\mathcal{O}}^\times(E)) \to H^2(A, \overline{\mathcal{O}}^\times)$$

By Proposition 4.5,

$$H^1(E,\overline{\mathcal{O}}^\times)=\mathrm{Pic}(\overline{B})[\tfrac{1}{p}]/M_2^\vee,$$

where by Definition 6.12, $M_2^\vee$ denotes the image of $M^\vee[\frac{1}{p}]\to B(k)[\frac{1}{p}]$, and where the right hand side are the locally constant maps.

It follows from Proposition 6.14 and Theorem 5.1 that the first, third and fourth term of the above sequence remain the same after taking $\varinjlim_{[N]}$, hence the same holds for $H^1(A,\overline{\mathcal{O}}^\times)$. This proves the following, which may already be regarded as a generalisation of Theorem 5.1:

Corollary 6.28. *The natural morphisms induce isomorphisms*

$$\mathrm{Pic}(\widetilde{A})=H^1(\widetilde{A},\overline{\mathcal{O}}^\times)=H^1(A,\overline{\mathcal{O}}^\times).$$

To prove the Theorem, it remains to show that we may replace δ_2 with the simpler term stated in the theorem. We do this by comparing to the rigid situation: By a comparison of Cartan–Leray sequences for the map $\mathcal{O}^\times\to\overline{\mathcal{O}}^\times$ and the morphism of M-torsors $X\to E$ over $\widetilde{A}\to A$, we get a commutative "master diagram"

$$\begin{array}{ccccccccc}
0 \to & H^1(M,\overline{\mathcal{O}}^\times(E)) & \to & H^1(A,\overline{\mathcal{O}}^\times) & \to & (\mathrm{Pic}(\overline{B})/M_2^\vee)^M & \xrightarrow{\delta_2} & \mathrm{Hom}(\wedge^2 M, K^\times/(1+\mathfrak{m})) \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 \to & H^1(M,\mathcal{O}^\times(E)) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & (\mathrm{Pic}(B)/M^\vee)^M & \xrightarrow{\delta_2} & \mathrm{Hom}(\wedge^2 M, K^\times).
\end{array}$$

Let $N\in(\mathrm{Pic}(B)/M_2^\vee)^M$. By Proposition 6.18, we can without loss of generality assume after changing A without changing $\widetilde{A}$ that N is in the image of the third column, i.e. arises from some line bundle N_0 on B that maps into $(\mathrm{Pic}(B)/M_2^\vee)^M$.

Since δ_2 on the bottom row has the desired form by the rigid Appell–Humbert Theorem 6.4, this shows that δ_2 in the top line is as described. ☐

Theorem 6.29. *For any choice of basis $m_1,\dots,m_r$ of M there is a natural surjection onto $\mathrm{Pic}(\widetilde{A}_p)$ from the set of triples (N,λ,r) consisting of*

$$N\in\mathrm{Pic}(\overline{B}),\quad \lambda: M[\tfrac{1}{p}]\to M^\vee[\tfrac{1}{p}],\quad r_i: g(m_i)^*N\xrightarrow{\sim}K^\times/(1+\mathfrak{m}).$$

such that the following conditions holds:

(1) *The following diagram commutes*

$$\begin{array}{ccc}
M & \xrightarrow{g} & B(k) \\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\varphi_N} \\
M^\vee & \xrightarrow{g'} & B^\vee(k),
\end{array}$$

thus inducing a morphism $\varphi_E:\widetilde{E}\to\widetilde{E}'$.

(2) *The following diagram commutes up to topological torsion:*

$$\begin{array}{ccc}
M & \xrightarrow{g} & \widetilde{E} \\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\varphi_E} \\
M^\vee & \xrightarrow{g'} & \widetilde{E}^\vee.
\end{array}$$

Proof. We begin with a further reduction step:
Step 3: Reducing to totally non-degenerate and non-degenerate case. By Lemma 6.21, we can without loss of generality assume that the kernel of $\phi: M^\vee\to B^\vee(k)\otimes_{\mathbb{Z}}\mathbb{Q}$ coincides with the kernel of $\phi: M^\vee\to B^\vee(K)$. By Lemma 6.2.(3) and (4), we can then replace A by an isogeneous abeloid that admits a decomposition

$$A=A_1\times A_2$$

where $A_1 = T_1/M_1$ is the maximal totally degenerate quotient of A with associated Raynaud extension $E_1 = T_1$ satisfying $\mathcal{O}^\times(E_1) = M_1^\vee \times K^\times$. Moreover, we have $\overline{\mathcal{O}}^\times(E_1) = M_1^\vee[\frac{1}{p}] \times K^\times/(1+\mathfrak{m})$ and $\overline{\mathcal{O}}^\times(E_2) = K^\times/(1+\mathfrak{m})$. It follows from inflation-restriction that we have

$$H^1(M, \overline{\mathcal{O}}^\times(E)) = H^1(M_1, \overline{\mathcal{O}}^\times(E_1)) \times H^1(M_2, \overline{\mathcal{O}}^\times(E_2)).$$

Similarly, the transition morphism

$$\ker(\varinjlim_n (\mathrm{Pic}(\widetilde{B})/M_2^\vee)^{p^n M} \xrightarrow{\delta_2} \mathrm{Hom}(\wedge^2 M, K^\times/(1+\mathfrak{m})))$$

identifies with the analogous term for $\widetilde{A}_2$, since this is true in the rigid situation. It follows from comparing the Appell–Humbert sequences for $\widetilde{A}_1$ and $\widetilde{A}_2$ that we have

$$\mathrm{Pic}(\widetilde{A}) = \mathrm{Pic}(\widetilde{A}_1) \times \mathrm{Pic}(\widetilde{A}_2)$$

where A_1 is totally degenerate and A_2 is non-degenerate (this could also be seen directly by proving that there are no non-constant maps from $\widetilde{A}_2$ into the Picard functor of $\widetilde{A}_1$).

At this point, we may treat the totally degenerate and the non-degenerate case separately. Step 4: The totally degenerate case. In the totally degenerate case, we have $B = 0$ and thus the sequence is simply a commutative diagram

$$\begin{array}{ccccccc} 0 & \to & H^1(M, \overline{\mathcal{O}}^\times(E)) & \xrightarrow{\sim} & \mathrm{Pic}(\widetilde{A}) & \to & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \to & H^1(M, \mathcal{O}^\times(E)) & \xrightarrow{\sim} & \mathrm{Pic}(A) & \to & 0 \end{array}$$

as in the complex Appell–Humbert Theorem. By Lemma 6.13, we have a short exact sequence

$$0 \to K^\times/(1+\mathfrak{m}) \to \overline{\mathcal{O}}^\times(E) \to M^\vee[\tfrac{1}{p}] \to 0.$$

The long exact sequence of group cohomology for this is of the form

$$\begin{array}{ccccccccc} M^\vee[\frac{1}{p}] & \to & \mathrm{Hom}(M, K^\times/(1+\mathfrak{m})) & \to & H^1(M, \overline{\mathcal{O}}^\times(E)) & \to & \mathrm{Hom}(M, M^\vee[\frac{1}{p}]) & \xrightarrow{\delta_1} & \mathrm{Hom}(\wedge^2 M, K^\times/(1+\mathfrak{m})) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ M^\vee & \longrightarrow & \mathrm{Hom}(M, K^\times) & \longrightarrow & H^1(M, \mathcal{O}^\times(E)) & \longrightarrow & \mathrm{Hom}(M, M^\vee) & \xrightarrow{\delta_1} & \mathrm{Hom}(\wedge^2 M, K^\times). \end{array}$$

This shows that the left part of the sequence accounts for $\mathrm{Pic}^0(\widetilde{A})$ which in the totally degenerate case is given by

$$\mathrm{Pic}^0(\widetilde{A}) = T/\widehat{T}(K).$$

We conclude that the right part of the diagram induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{NS}(\widetilde{A}) & \longrightarrow & \mathrm{Hom}(M, M^\vee) & \xrightarrow{\delta_1} & \mathrm{Hom}(\wedge^2 M, K^\times/(1+\mathfrak{m})) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{NS}(A) & \longrightarrow & \mathrm{Hom}(M, M^\vee) & \xrightarrow{\delta_1} & \mathrm{Hom}(\wedge^2 M, K^\times) \end{array}$$

Let $\lambda : M \to M^\vee$ be an element in $\mathrm{NS}(\widetilde{A})$. Then the defining condition that $\lambda \in \ker \delta_1$ means that $\overline{\psi}_{\mathcal{O},\lambda} = 1$ where $\mathcal{O}$ is the trivial line bundle. By Lemma 6.22, this means that

$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ \downarrow & & \downarrow \\ T/\widehat{T}(k)[\frac{1}{p}] & \xrightarrow{\varphi} & T^\vee/\widehat{T}^\vee(k)[\frac{1}{p}] \end{array}$$

commutes, which by the same lemma is equivalent to the last statement in the theorem.

Finally, to see Corollary 6.25 in this case, we note that by Proposition 6.23, we can always change A without changing $\widetilde{A}$ to assume that λ is in the image of $\mathrm{NS}(A) \to \mathrm{NS}(\widetilde{A})$.

Step 5: The non-degenerate case. It remains to treat the non-degenerate case, which means precisely that $\overline{\mathcal{O}}^{\times}(E) = K^{\times}/(1+\mathfrak{m})$ and thus also $\mathcal{O}^{\times}(E) = K^{\times}$. In particular, the first terms in our "master diagram" simplify, and it becomes

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(M, K^{\times}/(1+\mathfrak{m})) & \to & \mathrm{Pic}(\widetilde{A}) & \to & (\mathrm{Pic}(\overline{B})/M_2^{\vee})^M & \xrightarrow{\delta_2} & \mathrm{Hom}(\wedge^2 M, K^{\times}/(1+\mathfrak{m})) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathrm{Hom}(M, K^{\times}) & \to & \mathrm{Pic}(A) & \to & (\mathrm{Pic}(B)/M^{\vee})^M & \xrightarrow{\delta_2} & \mathrm{Hom}(\wedge^2 M, K^{\times}). \end{array}$$

Suppose now that we have a pair (N, λ) defining an element in $\ker \delta_2$. We have already found $N_0 \in \mathrm{Pic}(B)$ such that this comes from the bottom row. We claim that we can in fact choose A in such a way that $\delta_2(N_0) = 0$. This will prove Corollary 6.25 in this case by a diagram chase using that the leftmost map in the diagram is now surjective.

To see this, we choose any trivialisation $r : M \to N_0$ over $M \to B'$ and use the explicit description in Theorem 6.24 of

$$\delta_2(N_0) = \big(m_1, m_2 \mapsto r(m_1+m_2)r(m_1)^{-1}r(m_2)^{-1}r(0)\langle m_1, \lambda(m_2)\rangle\big).$$

If this is sent to $1 \in H^2(M, K^{\times}/(1+\mathfrak{m}))$, this means that there is $s : M \to K^{\times}/(1+\mathfrak{m})$ such that we have

$$\cdots = s(m_1+m_2)s(m_1)^{-1}s(m_2)^{-1}.$$

This implies in particular that $\langle m_1, \lambda(m_2)\rangle$ is symmetric mod $1+\mathfrak{m}$; More precisely, unravelling the implicit identification

$$\overline{\mathcal{P}}_{m_1 \times \lambda(m_2)} \xrightarrow[\sim]{\alpha} \mathcal{D}(N)_{m_1,m_2} \cong \mathcal{D}(N)_{m_2,m_1} \xrightarrow[\sim]{\alpha^{-1}} \overline{\mathcal{P}}_{m_2 \times \lambda(m_1)},$$

which equals φ, this means that

$$M^2 \to K^{\times}/(1+\mathfrak{m}), \quad m_1, m_2 \mapsto \langle m_1, \lambda(m_2)\rangle \Big/ \varphi\big(\langle m_2, \lambda(m_1)\rangle\big)$$

vanishes. By Lemma 6.22, this means that the following diagram commutes

$$\begin{array}{ccc} M[\tfrac{1}{p}] & \xrightarrow{\lambda} & M^{\vee}[\tfrac{1}{p}] \\ \downarrow & & \downarrow \\ \overline{E}(k)[\tfrac{1}{p}] & \xrightarrow{\varphi} & \overline{E}^{\vee}(k)[\tfrac{1}{p}], \end{array}$$

where $\overline{E}$ and $\overline{E}'$ are the associated $\overline{\mathcal{O}}^{\times}$-torsors obtained via pushout along $\mathcal{O}^{\times} \to \overline{\mathcal{O}}^{\times}$. By Proposition 6.23, we can then choose A in such a way that already the lift

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^{\vee} \\ \downarrow & & \downarrow \\ E(K) & \xrightarrow{\varphi_{N'}} & E^{\vee}(K), \end{array}$$

commutes. This in turn implies that $\delta_2(N_0) = 1$, as we wanted to see.

Conversely, if $\langle m_1, \lambda(m_2)\rangle$ is symmetric, the lift we just obtained satisfies $\delta_2(N_0) = 1$ by Theorem 6.4, which implies $\delta_2(N) = 1$ by commutativity of the above comparison diagram. This shows that the symmetry of $\langle -, \lambda(-)\rangle : M \times M \to K^{\times}/(1+\mathfrak{m})$ is in fact equivalent to $\delta_2(N) = 1$. Applying Lemma 6.22.3, this is equivalent to the last statement in the theorem. This completes the proof of the theorem. ☐

6.8. **Picard group of the p-adic cover.**

Definition 6.30. We call a morphism $\varphi : \widetilde{A}_p \to \widetilde{A}_p^{\vee}$ symmetric if the induced morphism $\overline{\varphi} : \overline{B} \to \overline{B}^{\vee}$ on the special fibre of the good reduction part is symmetric, i.e. if $\overline{\varphi}^{\vee} = \overline{\varphi}$. The subset of symmetric morphisms is denoted by $\mathrm{Hom}(\widetilde{A}_p, \widetilde{A}_p^{\vee})^{\mathrm{sym}} \subseteq \mathrm{Hom}(\widetilde{A}_p, \widetilde{A}_p^{\vee})$.

We now get the following very close analogue of the rigid Theorem 6.9:

Theorem 6.31. *There is a natural short exact sequence*

$$0 \to A(K)/\widehat{A}(K) \to \mathrm{Pic}(\widetilde{A}_p) \to \mathrm{Hom}(\widetilde{A}_p, \widetilde{A}_p^\vee)^{\mathrm{sym}} \to 0.$$

Here the first term can be identified with $\mathrm{Pic}^0(\widetilde{A}_p) := \mathrm{Ext}^1(\widetilde{A}_p, \mathbb{G}_m)$ *and the third is by definition the Néron–Severi group* $\mathrm{NS}(\widetilde{A}_p)$*, which is the finite free* $\mathbb{Z}[\frac{1}{p}]$*-submodule*

$$\mathrm{NS}(\widetilde{A}_p) \subseteq \mathrm{NS}(\overline{B})[\tfrac{1}{p}] \times \mathrm{Hom}(M, M^\vee)[\tfrac{1}{p}]$$

of pairs (N, λ) *satisfying both of the following conditions:*

(1) The following diagram commutes (cf Lemma 6.21):

$$\begin{array}{ccc} M[\frac{1}{p}] & \xrightarrow{\lambda} & M^\vee[\frac{1}{p}] \\ {\scriptstyle \overline{\phi}}\downarrow & & \downarrow{\scriptstyle \overline{\phi}^\vee} \\ \overline{B}(k)[\frac{1}{p}] & \xrightarrow{\varphi_N} & \overline{B}^\vee(k)[\frac{1}{p}]. \end{array}$$

(2) The pairing $\langle -, \lambda(-)\rangle_N : M \times M \to \overline{\mathcal{P}}_{B\times B^\vee}$ *is symmetric (cf Lemma 6.22)*

Remark 6.32. The morphism on the right can be given a moduli interpretation as follows: Any line bundle on L on $\widetilde{A}_p$ defines a line bundle on $\widetilde{A}_p \times \widetilde{A}_p$ given by

$$m^*L \otimes \pi_2^* L^{-1}.$$

One can prove that a version of the Theorem of the Square holds on $\widetilde{A}_p$, which implies that this is translation invariant and thus defines a morphism $\widetilde{A}_p \to A^\vee/A^{\vee\mathrm{tt}}$. By [16, Corollary 3.11], this lifts uniquely to the desired map

$$\widetilde{A}_p \to \widetilde{A}_p^\vee.$$

However, proving the Theorem of the Square and exactness of the sequence in the Theorem, requires a lot of work; it is easier to deduce this directly from the statement that any line bundle on $\widetilde{A}_p$ comes from some abeloid.

Remark 6.33. Theorem 6.31 gives a precise answer to Question 1.1 for $\widetilde{A}_p \to A$. Namely, in comparison to the analogous description of $\mathrm{NS}(A)$, we see that there are in general three sources of additional line bundles on $\mathrm{Pic}(\widetilde{A}_p)$ that may be invisible in $\varinjlim_{[p]} \mathrm{Pic}(A)$:

- The Néron–Severi group of the special fibre $\mathrm{NS}(\overline{B})$ may be larger than $\mathrm{NS}(B)$.
- Given any line bundle N on B, the pullback to E needs to be M-invariant for N to descend to A. For descent from $\widetilde{E}_p$ to $\widetilde{A}_p$, condition 1 says that it suffices to be M-invariant on the special fibre, which is a weaker condition.
- Given any line bundle N on B, the cocycle condition might not be satisfied for descent along $E \to A$, but can still be satisfied for the analogous cover $X := M_p \times \widetilde{E}_p \to \widetilde{A}_p$ since here, condition 2 says that it suffices for the cocycle to vanish mod $1 + \mathfrak{m}$.

Proof. This follows from the second part of the perfectoid Appell–Humbert Theorem and the description of $\mathrm{Hom}(\widetilde{A}_p, \widetilde{A}_p^\vee)$ in [16]. □

Finally, we deduce the Main Theorem: The description of the Picard functor of $\widetilde{A}_p$:

Theorem 6.34. *There is a natural short exact sequence*

$$0 \to A/\widehat{A} \to \mathbf{Pic}_{\widetilde{A}_p} \to \underline{\mathrm{Hom}^{\mathrm{sym}}(\widetilde{A}_p, \widetilde{A}_p^\vee)} \to 0.$$

Proof. As before, via tilting, it suffices to prove this in characteristic 0. Here the result is an application of Theorem 2.4. □

Remark 6.35 (Polarisations)**.** By composing with the valuation $\log|-| : K^\times/(1+\mathfrak{m}) \to \mathbb{R}$, one can define for any $(N, \lambda) \in \mathrm{NS}(\widetilde{A}_p)$ an associated symmetric bilinear form

$$|\langle -, \lambda - \rangle_{N,\lambda}| : M[\tfrac{1}{p}] \times M[\tfrac{1}{p}] \to \mathbb{R}.$$

In particular, if the corresponding line bundle comes from A, one can recover on $\widetilde{A}_p$ the information whether this line bundle is ample: This is the case if and only if $|\langle -, \lambda - \rangle|$ is symmetric positive definite by [26, Theorem 6.4.6]. In this case, one can show by tilting that N sets up a morphism $\widetilde{A}_p \to \mathbb{P}^{n,\mathrm{perf}}$ for some n that is a pro-finite-étale morphism composed with a closed immersion.

The locally constant sheaf in Theorem 6.34 can also be written as an inner Hom sheaf, like for abelian varieties. In fact, this can actually be deduced from the Theorem itself:

Corollary 6.36. *Let A and A' be two abeloids. Then we have*

$$\mathbf{Hom}(\widetilde{A}_p, \widetilde{A}'_p) = \underline{\mathrm{Hom}(\widetilde{A}_p, \widetilde{A}'_p)}.$$

Proof. Let S be any affinoid perfectoid space, we need to show that every homomorphism

$$f : \widetilde{A}_p \times S \to \widetilde{A}'_p$$

is locally constant in S, i.e. there is a disjoint analytic cover of S on which the map factors through the forgetful map $\widetilde{A}_p \times S \to \widetilde{A}_p$. Let thus S-linear homomorphism $\widetilde{A}_p \times S \to \widetilde{A}'_p$. It suffices to prove the statement for the projection

$$\widetilde{A}_p \times S \to A'.$$

This in turn corresponds to a line bundle L on $\widetilde{A}_p \times A'^\vee \times S$ that is translation invariant on the second factor and trivial after specialisation at the identity of either the first or second factor. Considering the short exact sequence

$$0 \to \widehat{A}'(S) \to \mathrm{Pic}_{\text{ét}}(\widetilde{A}_p \times A'^\vee \times S) \to \mathrm{Pic}_{\text{ét}}(\widetilde{A}_p \times \widetilde{A}'^\vee_p \times S),$$

we conclude that it suffices to consider the pullback of L to $\widetilde{A}_p \times \widetilde{A}'^\vee_p \times S$. But by the Theorem we have a short exact sequence

$$0 \to A/\widehat{A}(S) \times A'^\vee/\widehat{A}'^\vee(S) \to \mathbf{Pic}_{\widetilde{A}_p \times \widetilde{A}'^\vee_p}(S) \to \underline{\mathrm{Hom}(\widetilde{A}_p \times \widetilde{A}'^\vee_p, \widetilde{A}^\vee_p \times \widetilde{A}'_p)}(S) \to 0.$$

The translation invariance in $\widetilde{A}'^\vee_p$ and the fact that the pullback to $\widetilde{A}_p$ is trivial mean that the image of L in the last term is already contained in

$$\underline{\mathrm{Hom}(\widetilde{A}_p, \widetilde{A}'_p)}(S).$$

This defines a morphism $g : \widetilde{A}_p \times S \to \widetilde{A}'_p$ that is locally constant in S. Since f and g then define line bundles whose image in the third term coincide, the difference $f - g$ defines a line bundle on $\widetilde{A}_p \times \widetilde{A}'_p \times \times S$ corresponding to an element in $A/\widehat{A}(S) \times A'^\vee/\widehat{A}'^\vee(S)$, the first component of which necessarily vanishes. The second defines an element in $S \to \widetilde{A}'_p$. The assumption that f is a homomorphism forces this to vanish as well. Thus $f = g$. □

6.9. **Picard group of the universal cover.** It is easy to deduce from the results for the pro-p-cover $\widetilde{A}_p$ the analogous results for the pro-étale universal cover

$$\widetilde{A} = \varprojlim_{[n]} A \to A,$$

where n ranges over all of $n \in \mathbb{N}$:

Theorem 6.37. *There is a natural isomophism* $\mathrm{Pic}(\widetilde{A}) = \mathrm{Pic}(\widetilde{A}_p) \otimes \mathbb{Q}$. *In particular, there is a natural short exact sequence*

$$0 \to A^\vee(K)/A^\vee(K)^{\mathrm{tt}} \to \mathrm{Pic}(\widetilde{A}) \to \mathrm{Hom}(\widetilde{A}, \widetilde{A}^\vee)^{\mathrm{sym}} \to 0.$$

Here the first term can be identified with $\mathrm{Pic}^0(\widetilde{A}) := \mathrm{Ext}^1(\widetilde{A}, \mathbb{G}_m)$ *and the third is the group* $\mathrm{NS}(\widetilde{A}) := \mathrm{Pic}(\widetilde{A})/\mathrm{Pic}^0(\widetilde{A})$, *which is a finite dimensional $\mathbb{Q}$-vector space.*

We obtain the following very close direct analogue of the Appell–Humbert Theorem:

Corollary 6.38. *Assume that $K = \mathbb{C}_p$ or $K = \mathbb{C}_p^\flat$. Then*

$$\mathrm{Pic}(\widetilde{A}) = \mathrm{Hom}(\widetilde{A}, \widetilde{A}^\vee)^{\mathrm{sym}}$$

is a finite dimensional $\mathbb{Q}$-vector space of dimension $\geq \mathrm{rk}(\mathrm{NS}(A))$ isomorphic to the subspace

$$\mathrm{NS}(\widetilde{A}) \subseteq (\mathrm{NS}(\overline{B}) \times \mathrm{Hom}(M, M^\vee)) \otimes \mathbb{Q}$$

of pairs (N, λ) such that the pairing $|\langle -, \lambda(-)\rangle_{N,\lambda}| : M \times M \to \mathbb{Q}$ is symmetric.

Proof. We consider the case of $\mathbb{C}_p$, the case of $\mathbb{C}_p^\flat$ is completely analogous, and also follows via tilting.

The first part follows from Theorem 6.37 since $A^\vee(\mathbb{C}_p)^{\mathrm{tt}} = \mathbb{A}^\vee(\mathbb{C}_p)$ by [18, Proposition 2.23].

The second part follows from tensoring the second part of Theorem 6.31 with $\mathbb{Q}$. Here we use that after tensoring with $\mathbb{Q}$, the first condition becomes vacuous since $B^\vee(\overline{\mathbb{F}}_p)$ is a torsion group, and the term $\mathbb{C}_p^\times/(1+\mathfrak{m})$ becomes isomorphic to $\mathbb{Q}$ via the p-adic valuation. □

Proof of Theorem 6.37. We have

$$\mathrm{Pic}(\widetilde{A}) = H^1(A, \mathcal{O}^\times/\mathcal{O}^{\times,\mathrm{tt}}) = H^1(A, \mathcal{O}^\times/\mathcal{O}^{\times,\mathrm{tt}}) \otimes \mathbb{Q} = \mathrm{Pic}(\widetilde{A}) \otimes \mathbb{Q}.$$

The statement thus follows from tensoring the sequence in Theorem 6.31 with $\mathbb{Q}$, and the fact that the natural map

$$\mathrm{Hom}(\widetilde{A}_p, \widetilde{A}_p^\vee) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Hom}(\widetilde{A}, \widetilde{A}^\vee)$$

is an isomorphism by [16, Lemma 5.11]. □

7. Application: Tilting abelian varieties

Given an abeloid variety A over K, one can show that there is always an abeloid variety A' over $K^\flat$ for which there is an "untilting map" of diamonds

$$\sharp : A'^\diamondsuit \to A^\diamondsuit.$$

The abeloid variety A' is not unique and depends on certain choices. The goal of this section is to show that if A is an abelian variety, we can always arrange for A' to be an abelian variety, too. We will give an explicit construction of an abelian variety A' with such a sharp mp in terms of the Raynaud uniformisation of A. The strategy to show that A' is an abelian variety is to show that polarisations can be lifted along $\sharp$.

We begin with some recollections. Let A' be any abeloid variety over $K^\flat$. Then the diamond $A'^\diamondsuit$ is represented by the perfection

$$A'^{\mathrm{perf}} = \varprojlim_F A.$$

This is a perfectoid space over $K^\flat$ and we can form its untilt $A'^{\mathrm{perf}\sharp} \to \mathrm{Spa}(K)$. The main result of this section is:

Theorem 7.1. *Let A be an abelian variety over K. Then there is an abelian variety A' over $K^\flat$ with a pro-finite-étale cover*

$$A'^{\mathrm{perf}\sharp} \to A.$$

This is closely related to Wear's thesis, especially [36, Theorem 5.4.10] which shows the analogous result for the p-adic universal cover $\widetilde{A}'_p = \varprojlim_{[p]} A$ instead of A^{perf} (under an additional assumption on the value group of K to be $\mathbb{Q}$). Wear's approach uses formal models. We here give a slightly different approach, which needs no assumption on the base field, and instead relies on results and ideas from the previous sections in various ways.

7.1. **The case of good reduction.** For the proof of Theorem 7.1, we begin with some reduction steps. The following classical result shows that we can without loss of generality assume that A is principally polarised. Let us fix a level N coprime to p and let X denote the moduli space of principally polarised abelian varieties of dimension d with $\Gamma(N)$-level structure, considered as a rigid space. For a fixed $0 < \epsilon < \frac{1}{2}$, we denote by $X(\epsilon) \subseteq X$ the rigid analytic open subspace defined by the ϵ-ordinary locus in the sense of [35, §3].

Lemma 7.2. *Let A be an abelian variety of dimension d in characteristic 0 and let N be an ample line bundle on A.*

(1) There is an isogeny $A \to A_2$ such that N descends to a line bundle N_2 on A_2 and $\varphi_{N_2} : A_2 \to A_2^\vee$ is a principal polarisation.
(2) We can after a further isogeny assume moreover that A_2 admits a $\Gamma(N)$-level structure γ such that the data $(A, \varphi_{N_2}, \gamma)$ define a point in $X(\epsilon)$.

Proof. The first part is [28, Corollary 1 in §23].

For the second part, let γ be any $\Gamma(N)$-level structure, and let γ_{p^∞} be a $\Gamma(p^\infty)$-level structure. These exist because K is algebraically closed. Then $(A, \varphi_{N_2}, \gamma, \gamma_{p^\infty})$ defines a K-point in Scholze's infinite level moduli space $X_{\Gamma(p^\infty)}$ from [35, §3.3.2]. By [35, Lemma 3.3.11], we can after a further isogeny corresponding to the action of an element in $\mathrm{GSp}_{2d}(K)$ arrange that this point lies in $X_{\Gamma(p^\infty)}(\epsilon)$. ☐

Proposition 7.3. *Theorem 7.1 holds when $A = B$ has good reduction. Moreover, given a principal polarisation $\lambda : B \to B^\vee$, there exists a unique principal polarisation $\lambda' : B' \to B'^\vee$ such that the following diagram commutes in the category of diamonds*

$$\begin{array}{ccc} B' & \xrightarrow{\lambda'} & B'^\vee \\ \downarrow{\scriptstyle\sharp} & & \downarrow{\scriptstyle\sharp} \\ B & \xrightarrow{\lambda} & B^\vee. \end{array}$$

Proof. By Lemma 7.2, we may assume that B defines a K-point of $\mathcal{X}(\epsilon)$, the good reduction locus inside $X(\epsilon)$. We now recall some results from [35, §3.2.3] about tilting moduli spaces of good reduction: By [35, Corollary III.2.19] The moduli space $\mathcal{X}(\epsilon)$ admits a pro-étale affinoid perfectoid cover

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}(\epsilon),$$

called the anticanonical tower. Over it, there is a canonical tower of universal abelian varieties

$$\mathcal{A}_{\Gamma_0(p^\infty)}(\epsilon) = \varprojlim_{n\in\mathbb{N}} (\mathcal{A}_{\Gamma_0(p^n)}(\epsilon) \times_{\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a} \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a)$$

which is also perfectoid. The limit of the universal polarisations defines a canonical isomorphism

$$\lambda : \mathcal{A}_{\Gamma_0(p^\infty)}(\epsilon) \to \mathcal{A}^\vee_{\Gamma_0(p^\infty)}(\epsilon)$$

where the right hand side denotes the induced tower of dual abelian varieties. Finally, [35, Corollary III.2.19] says that for the tilts, we have canonical identifications

$$\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a^\flat = \mathcal{X}'(\epsilon)^{\mathrm{perf}}, \quad \mathcal{A}_{\Gamma_0(p^\infty)}(\epsilon)^\flat = \mathcal{A}_{\Gamma_0(p^\infty)}(\epsilon)^{\mathrm{perf}}.$$

The proof of [35, Corollary III.2.19] applies also to the dual abelian variety, yielding an identification of the universal polarisations.

The Proposition now follows from choosing any lift of the point $\mathrm{Spa}(K) \to \mathcal{X}(\epsilon)$ to $\mathrm{Spa}(K) \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (which exists since K is algebraically closed) and taking the fibre of the above tilting isomorphisms over this point. ☐

Let us denote

$$B^{\mathrm{perf}} := B'^{\mathrm{perf}\sharp}.$$

Here we need to keep in mind that $-^{\mathrm{perf}}$ is not completely functorial in this context because it depends on a choice of level structure at p.

Our next goal is to compare line bundles on B^{perf} and B'^{perf}. As we will shortly recall, the group $\mathrm{Pic}(B'^{\mathrm{perf}})$ has been computed in [15]: It sits in a left-exact sequence

$$0 \to \mathrm{Hom}(T_pB, K^\times) \to \mathrm{Pic}(B^{\mathrm{perf}}) \to \mathrm{Pic}(\overline{B})[\tfrac{1}{p}]^{T_pB} \tag{26}$$

where $\overline{B}$ is the special fibre. This means that in general, the size of image of $B^\vee(K) \to \mathrm{Pic}(B^{\mathrm{perf}})$ depends on the size of the Tate module and can be anywhere in between $B^\vee(K)[\frac{1}{p}]$ (when B is ordinary) and $\overline{B}^\vee(k)[\frac{1}{p}]$ (when B is supersingular).

In fact, the sequence (26) can easily be upgraded to a sequence of Picard functors:

Lemma 7.4. *We have a morphism of left-exact sequences of Picard functors defined on* $\mathrm{Perf}_K = \mathrm{Perf}_{K^\flat}$*, where in the top line we use* $\mathbb{G}_m^\flat$ *whereas in the bottom line we use* $\mathbb{G}_m$*.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{Hom}(T_pB', \mathbb{G}_m^\flat) & \longrightarrow & \mathbf{Pic}_{B'^{\mathrm{perf}}} & \longrightarrow & \mathbf{Pic}_{\widetilde{B}'_p} \\
 & & \downarrow{\scriptstyle\sharp} & & \downarrow{\scriptstyle\sharp} & & \downarrow{\scriptstyle\sharp} \\
0 & \longrightarrow & \mathbf{Hom}(T_pB', \mathbb{G}_m) & \longrightarrow & \mathbf{Pic}_{B^{\mathrm{perf}}} & \longrightarrow & \mathbf{Pic}_{\widetilde{B}_p}.
\end{array}$$

The left vertical map is surjective in the pro-étale topology, the right vertical map is an isomorphism.

Proof. The diagram arises from the Cartan–Leray sequence, using that both $\widetilde{B}'_p \to B'^{\mathrm{perf}}$ and $\widetilde{B}_p \to X$ are pro-étale torsors under T_pB', which are identified via tilting.

The left map is surjective because $\sharp : \widehat{\mathbb{G}}_m^\flat \to \widehat{\mathbb{G}}_m$ is surjective. The rightmost map is an isomorphism by Corollary 6.16. ☐

Lemma 7.5. *In the setting of Proposition 7.3, the following diagram commutes:*

$$\begin{array}{ccccc}
B'^\vee(K^\flat) & = & \mathrm{Pic}^0(B') & \longrightarrow & \mathrm{Pic}(B'^{\mathrm{perf}}) \\
\downarrow{\scriptstyle\sharp} & & & & \downarrow{\scriptstyle\sharp} \\
B^\vee(K) & = & \mathrm{Pic}^0(B) & \longrightarrow & \mathrm{Pic}(B^{\mathrm{perf}}).
\end{array}$$

Proof. It is clear that the diagram upgrades to the respective Picard functors on $\mathrm{Perf}_K = \mathrm{Perf}_{K^\flat}$. A a first step, we claim that the following diagram commutes:

$$\begin{array}{ccccc}
B'^\vee & = & \mathbf{Pic}_{B'} & \longrightarrow & \mathbf{Pic}_{\widetilde{B}'_p} \\
\downarrow{\scriptstyle\sharp} & & & & \downarrow{\scriptstyle\sharp} \\
B^\vee & = & \mathbf{Pic}_{B} & \longrightarrow & \mathbf{Pic}_{\widetilde{B}_p}.
\end{array}$$

To see this, we recall from [15, §5] that the top horizontal line factors through the diamantine reduction map

$$B'^\vee \to \overline{B}'^{\vee\diamondsuit}$$

and similarly for the bottom map. The commutativity then follows from the fact that $\sharp$ commutes with reductions to k by construction.

Consider now the difference $\delta : B'^\vee \to \mathbf{Pic}_{B^{\mathrm{perf}}}$ between the morphism $B'^\vee \to \mathbf{Pic}_{B'^{\mathrm{perf}}} \to \mathbf{Pic}_{B^{\mathrm{perf}}}$ and $B'^\vee \to B^\vee \to \mathbf{Pic}_{B^{\mathrm{perf}}}$. By comparing to Lemma 7.4, it follows that δ factors through a morphism

$$B'^\vee \to \mathbf{Hom}(T_pB', \mathbb{G}_m).$$

But any morphism $B'^\vee \to \mathbb{G}_m$ is trivial, hence $\delta = 0$. ☐

7.2. **Perfections of Raynaud extensions.** As before, let A be an abeloid variety with principal polarisation $\varphi : A \to A^\vee$. Let $T \to E \to B$ and $A = E/M$ and $\phi : M^\vee \to B^\vee(K)$ be the associated data. Let L be an ample line bundle on A corresponding to φ and let N be any line bundle on B whose pullback to E identifies with that of L.

Our next goal is to lift the morphism $\sharp : B'^\vee \to B^\vee$ to Raynaud extensions. For this we first need to identify the correct Raynaud extension of B' which will be related to E via tilting. We will find it by way of the following variant of Proposition 6.18:

Proposition 7.6. *Let $\lambda : M \to M^\vee$ be a homomorphism such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^\vee} \\ B(K) & \xrightarrow{\varphi_N} & B^\vee(K). \end{array}$$

Such a diagram is associated to the datum of L. Then after replacing M by a subgroup of the same rank, we can find $\phi' : M \to B'(K^\flat)$ and $\phi'^\vee : M^\vee \to B'^\vee(K^\flat)$ with $\phi = \sharp \circ \phi'$ and $\phi^\vee = \sharp \circ \phi'^\vee$ making the following lift of this diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi'^\vee} \\ B'(K^\flat) & \xrightarrow{\varphi_{N'}} & B'^\vee(K^\flat)[\frac{1}{p}]. \end{array}$$

Moreover, we can arrange that $\ker \overline{\phi}^\vee = \ker \phi'^\vee$.

Proof. We first choose any lifts $\phi' : M \to B'(K^\flat)$ and $\phi'^\vee : M^\vee \to B'^\vee(K^\flat)$ along the sharp maps. We then apply Lemma 6.19 with $U(B') := \ker(B'(K^\flat) \to B(K) \otimes \mathbb{Q})$ and $U(B') := \ker(B'^\vee(K^\flat) \to B^\vee(K) \otimes \mathbb{Q})$, to find ϕ' and $\phi'^\vee$ making the diagram commute up to torsion. After replacing M and $M^\vee$ by subgroups corresponding to an isogeny $A'' \to A'$, the diagram commutes. ☐

The morphism $\phi'^\vee$ induces a Raynaud extension

$$0 \to T' \to E' \to B' \to 0$$

where T' is the analytic torus over $K^\flat$ with character group $M^\vee$. We consider its perfection

$$0 \to T'^{\mathrm{perf}} \to E'^{\mathrm{perf}} \to B'^{\mathrm{perf}} \to 0.$$

Lemma 7.7. *There is a natural morphism*

$$E'^{\mathrm{perf}\sharp} \to E \times_B B^{\mathrm{perf}}$$

which can be characterised as the pushout of $E'^{\mathrm{perf}\sharp}$ along the canonical map $T'^{\mathrm{perf}\sharp} \to T$.

Proof. The extension E'^{perf} is determined by its class in $\mathrm{Ext}^1(B'^{\mathrm{perf}}, T'^{\mathrm{perf}})$ which can be described as the composition

$$M^\vee \xrightarrow{\phi'^\vee} B'^\vee(K^\flat) \to \mathrm{Pic}(B'^{\mathrm{perf}}).$$

The pushout of $E'^{\mathrm{perf}\sharp}$ along $T'^{\mathrm{perf}\sharp} \to T$ is thus computed by the composition

$$M^\vee \xrightarrow{\phi'^\vee} B'^\vee(K^\flat) \to \mathrm{Pic}(B'^{\mathrm{perf}}) \xrightarrow{\sharp} \mathrm{Pic}(B^{\mathrm{perf}}).$$

By Lemma 7.5, this agrees with

$$M^\vee \xrightarrow{\phi'^\vee} B'^\vee(K^\flat) \xrightarrow{\sharp} B^\vee(K) \to \mathrm{Pic}(B^{\mathrm{perf}}).$$

The statement now follows from the fact that $\phi = \sharp \circ \phi'^\vee$ by construction in Proposition 7.6. ☐

Let us denote

$$E^{\mathrm{perf}} := E'^{\mathrm{perf}\sharp}.$$

By construction, this is an extension

$$0 \to T^{\mathrm{perf}} \to E^{\mathrm{perf}} \to B^{\mathrm{perf}} \to 0$$

covering $T \to E \to B$. We thus obtain a natural morphism

$$\sharp : E'^{\diamondsuit} = E'^{\mathrm{perf}} \xrightarrow{\sharp} E^{\mathrm{perf}} \to E.$$

By applying the Lemma to $E^\vee$, we obtain an analogous morphism

$$\sharp : E'^{\vee} \to E^{\vee}.$$

We can now obtain a generalisation of the diagram in Proposition 7.3 to Raynaud extensions:

Lemma 7.8. *Let $\varphi : E \to E^\vee$ and $\varphi' : E' \to E'^\vee$ be the morphisms corresponding to the diagrams in Proposition 7.6 via Theorem 6.4. Then the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{\varphi'} & E'^\vee \\ \downarrow \sharp & & \downarrow \sharp \\ E & \xrightarrow{\varphi} & E^\vee. \end{array}$$

Proof. By definition of the $\sharp$-map, it suffices to prove that the following diagram commutes:

$$\begin{array}{ccc} E'^{\mathrm{perf}} & \xrightarrow{\varphi'} & E'^{\vee\mathrm{perf}} \\ \downarrow \sharp & & \downarrow \sharp \\ E \times_B B^{\mathrm{perf}} & \xrightarrow{\varphi} & E^\vee \times_{B^\vee} B^{\vee\mathrm{perf}} \end{array}$$

where the bottom row is over the natural map $\varphi_N : B^{\mathrm{perf}} \to B^{\vee\mathrm{perf}}$. To see this, it suffices to compare pushouts along different morphisms $T^\vee \to \mathbb{G}_m$, corresponding to elements $m \in M$. Let us denote by $\mathrm{Pic}^0(B^{\mathrm{perf}})$ the image of the natural map $B^\vee = \mathrm{Pic}^0(B) \to \mathrm{Pic}(B^{\mathrm{perf}})$, and similarly for $B^{\vee\mathrm{perf}}$ etc. Then it therefore suffices to prove that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\lambda^\vee} & M^\vee \\ \downarrow \phi' & & \downarrow \phi'^\vee \\ \mathrm{Pic}^0(B'^{\vee\mathrm{perf}}) & \xrightarrow{\varphi'_N} & \mathrm{Pic}^0(B'^{\mathrm{perf}}) \\ \downarrow \sharp & & \downarrow \sharp \\ \mathrm{Pic}^0(B^{\vee\mathrm{perf}}) & \xrightarrow{\varphi_N} & \mathrm{Pic}^0(B^{\mathrm{perf}}). \end{array}$$

The bottom square commutes, this follows from Proposition 7.3 and Lemma 7.5 applied once to B and once to $B^\vee$. The top square commutes by Proposition 7.6. ☐

7.3. **Construction of A': The undegenerated case.** We continue with the previous setup: Let L be an ample line bundle on A. Let N be a line bundle on B compatible with A and $\lambda : M \to M^\vee$ and $\varphi : E \to E^\vee$ be the induced morphisms, as before.

In the previous section, we constructed first an abelian variety B' over $K^\flat$ and then a Raynaud extension $T' \to E' \to B'$ which is related to E via tilting. As a final step, we now lift $M \to E$ to a morphism $M \subseteq E'$ defining the desired abeloid A'. This is achieved by the following, which simultaneously lifts a part of the polarisation data:

Proposition 7.9. *After replacing M by a sublattice of the same rank, there exists lifts $M \to E'$ of $M \to B'$ and $M^\vee \to E'^\vee$ of $M^\vee \to E'^\vee(K)$ making the following diagram commute:*

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & M^\vee \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h'^\vee} \\ E' & \xrightarrow{\varphi'} & E'^\vee \\ \downarrow{\scriptstyle \sharp} & & \downarrow{\scriptstyle \sharp} \\ E & \xrightarrow{\varphi} & E^\vee \end{array} \qquad (h = \sharp \circ h' : M \to E,\quad h^\vee = \sharp \circ h'^\vee : M^\vee \to E^\vee)$$

and such that $A' := E'/M$ and $E'^\vee/M^\vee$ are dual abeloids over $K^\flat$ and φ' is symmetric.

Proof. We apply Proposition 6.23 to the top diagram with $E'(K) \to E(K) \otimes \mathbb{Q}$ playing the role of "$E(K) \to \overline{E}(K)$" in the statement of the Proposition, and the subgroup

$$L := \ker(\sharp : K^{\flat\times} \to K^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \subseteq K^{\flat\times}.$$

This produces lifts h' and $h'^\vee$ such that the diagram commutes up to torsion. Hence, after replacing M by a sublattice, the diagram commutes.

That φ' is symmetric follows from [26, Proposition 6.4.1.(c)]. □

Lemma 7.10. *In the situation of Proposition 7.9, the following diagram comutes:*

$$\begin{array}{ccc} A'^\vee(K^\flat) & \longrightarrow & \mathrm{Pic}(A'^{\mathrm{perf}}) \\ \downarrow{\scriptstyle \sharp} & & \downarrow{\scriptstyle \sharp} \\ A^\vee(K) & \longrightarrow & \mathrm{Pic}(A^{\mathrm{perf}}) \end{array}$$

Proof. This follows from Lemma 7.5 by comparing the Cartan–Leray sequence of $E^{\mathrm{perf}} \to A^{\mathrm{perf}}$ to that of $E \to A$. □

We can now prove the Main result of this section:

Theorem 7.11. *Let A be an abelian variety over K and let L be an ample line bundle on A. Then there exists an abelian variety A' with a pro-finite-étale $q : A^{\mathrm{perf}} := A'^{\mathrm{perf}\sharp} \to A$ and an ample line bundle L' on A' such that for the pullback map $q' : A'^{\mathrm{perf}} \to A'$ and the untilt map $\sharp : \mathrm{Pic}(A'^{\mathrm{perf}}) \to \mathrm{Pic}(A^{\mathrm{perf}})$, we have*

$$(q'^* L)^\sharp \cong q^* L.$$

The induced polarisations make the following diagram of diamonds commute:

$$\begin{array}{ccc} A' & \xrightarrow{\varphi_{L'}} & A'^\vee \\ \downarrow{\scriptstyle \sharp} & & \downarrow{\scriptstyle \sharp} \\ A & \xrightarrow{\varphi_L} & A^\vee. \end{array}$$

Proof. We are free to replace A by an isogeneous abelian variety. Hence we can assume by Lemma 7.2 that A admits a principal polarisation. By Theorem 6.9, this corresponds to a pair $(\lambda : M \to M^\vee, N \in \mathrm{Pic}_B)$ such that φ_N is a polarisation and the diagram in Proposition 7.6 commutes. By the conclusion of this Proposition and Proposition 7.9, we obtain an abelian variety A' over $K^\flat$ with a symmetric morphism

$$\varphi' : A' \to A'^\vee$$

making the desired diagram commute. It is clear from the construction of A' that we have natural morphisms $A'^{\mathrm{perf}\sharp} \to A$ and $A'^{\vee\mathrm{perf}\sharp} \to A^\vee$ making the diagram commute.

It remains to find L'. For this we note that by Theorem 6.9, the symmetric morphism φ' comes from *some* line bundle L'' on A'. By Theorem 6.31, we see that the difference between $(q'^* L'')^\sharp$ and L comes from a line bundle $L_0 \in A^\vee(K)$. By Lemma 7.10, we can lift this via $\sharp$ to a line bundle L_0', then $L' := L'' \otimes L_0'^{-1}$ satisfies $(q'^* L)^\sharp \cong q^* L$.

It remains to see that L' is ample, this will imply that A' is an abelian variety. But we can use the isomorphism $(q'^*L)^\sharp \cong q^*L$ to identify the pairings in [26, Theorem 6.4.4]. The fact that the pairing is positive definite for L thus implies the same statement for L', which implies that L' is ample.

□

7.4. **Example: The totally degenerate case.** To illustrate the theorem, we now give a more explicit exposition in the totally degenerate case. Hence we assume that $E = T$ and

$$A = T/M.$$

If K has characteristic p, the sequence $0 \to M \to T \to A \to 0$ induces by functoriality of the perfection a sequence

$$0 \to M \to T^{\mathrm{perf}} \to A^{\mathrm{perf}} \to 0.$$

In characteristic 0, we have an analogous construction [find ref in [3]]: For any $n \in \mathbb{N}$, starting with $M_0 = M$, we inductively choose compatible lattices $M_n \subseteq T$ that map isomorphically onto M_{n-1} under $[p] : T \to T$. In the limit over n, we obtain a perfectoid tilde-limit

$$A^{\mathrm{perf}} \sim \varprojlim_{[p]} T/M_n$$

which by construction sits in an exact sequence

$$0 \to M \to T^{\mathrm{perf}} \to A^{\mathrm{perf}} \to 0$$

where we denote $T^{\mathrm{perf}} := \widetilde{T}$.

Proposition 7.12. *Let M' be the image of the map $M \to T^{\mathrm{perf}}(K) \xrightarrow{\flat} T'(K^\flat)$. Then $A' := T'/M'$ is an abeloid variety and we have a natural morphism of diamonds*

$$\sharp : A' \to A.$$

Any line bundle on A lifts to a line bundle on $A'^{(p^n)}$ for some n large enough. In particular, if A is an abelian variety, then A' is an abelian variety, too.

Proof. Since the morphism $K^{\flat\times} \to K^\times$ is compatible with valuations, the morphism $T^{\mathrm{perf}}(K) \xrightarrow{\flat} T'(K^\flat)$ commutes with the maps $T^{\mathrm{perf}}(K) \to \mathrm{Hom}(M, \mathbb{R})$ and $T'^{\mathrm{perf}}(K^\flat) \to \mathrm{Hom}(M, \mathbb{R})$. Hence $A' = T'/M'$ is an abeloid variety. The morphism $\sharp : A' \to A$ is induced by the morphism $\sharp : T' \to T$ by considering short exact sequences.

It remains to see that we can tilt polarisations. For this we use:

Lemma 7.13. *The following pullback map is trivial*

$$\mathrm{Pic}(A^{\mathrm{perf}}) \to \mathrm{Pic}(T^{\mathrm{perf}}).$$

In contrast to the map $\mathrm{Pic}(A) \to \mathrm{Pic}(T)$, this is not immediately clear since by Theorem 4.17, the Picard group of T^{perf} is non-trivial.

Proof. We compare to the universal cover $\widetilde{A}$ which by [] admits a canonical analytic cover $M_p \times \widetilde{T} \to \widetilde{A}$ that fits into a diagram

$$\begin{array}{ccc} M_p \times T^{\mathrm{perf}} & \longrightarrow & \widetilde{A} \\ \big\downarrow & & \big\downarrow \\ T^{\mathrm{perf}} & \longrightarrow & A^{\mathrm{perf}} \end{array}$$

Pullback along the top map is trivial, this follows from Theorem 6.25 by comparing to $T \to A$. But the left vertical map is split. □

Proposition 7.14. *Let $A' = T/M$ be a totally degenerated abeloid. Then*

$$\mathrm{Pic}(A'^{\mathrm{perf}}) = H^1(M, \mathcal{O}^\times(T^{\mathrm{perf}})).$$

In particular, the natural morphism

$$\varinjlim_{F} \mathrm{Pic}(A') \to \mathrm{Pic}(A'^{\mathrm{perf}})$$

is an isomorphism.

We note that this is in contrast to the case of good reduction, where this map is in general neither injective nor surjective. This complements results of [15].

Proof. The statement thus follows from the Cartan–Leray sequence of $T^{\mathrm{perf}} \to A^{\mathrm{perf}}$ and Lemma 7.13. □

Returning to the proof of Proposition 7.12, let L be an ample line bundle on T corresponding to a polarisation $A \to A^\vee$. Then via pullback, we obtain a line bundle on A'^{perf}. By Proposition 7.14, this descends to a line bundle L' on $A'^{(p^n)}$ for some n. Since the associated bilinear pairing

$$M \times M^\vee \to K^{\flat\times}$$

is clearly related to the one of L via tilting, we see that L' is ample. Hence $A'^{(p^n)}$ is an abelian variety, which implies that so is A'. □

Appendix A. Computations in group cohomology

This appendix collects some group cohomologies that we need for the computations of the Picard group of universal covers of abeloids via descent from their analytic covers.

A.1. **$\mathbb{Z}^n$-actions: generalities.** For the last part of the proof, we used the following lemmas on group cohomology:

Lemma A.1. *Let G be any abelian group, endowed with the trivial M-action. Then*

$$H^n(M,G) = \wedge^n \mathrm{Hom}(M,G) = \mathrm{Hom}(\wedge^n M, G).$$

If G is a ring, this identifies the cup product on the left with the wedge-product on the right.

Proof. For $G = \mathbb{Z}$, this follows from the singular cohomology of the real torus

$$H^*((M \otimes_{\mathbb{Z}} \mathbb{R})/M, \mathbb{Z}) = \wedge_{\mathbb{Z}} \mathrm{Hom}(M,\mathbb{Z}) = \mathrm{Hom}(\wedge_{\mathbb{Z}} M, \mathbb{Z})$$

which identifies the cup product with the wedge product. Here to see the last isomorphism, we note that the cup product induces a natural map

$$\wedge^n_{\mathbb{Z}} \mathrm{Hom}(M,\mathbb{Z}) \to \mathrm{Hom}(\wedge^n_{\mathbb{Z}} M, \mathbb{Z}) \tag{27}$$

$$f_1 \wedge \dots \wedge f_n \mapsto \big(m_1 \wedge \dots \wedge m_n \mapsto \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma) f_1(m_{\sigma(1)}) \wedge \dots \wedge f_n(m_{\sigma(n)})\big). \tag{28}$$

In terms of any basis $M \cong \mathbb{Z}^r$, this sends $e_{i_1}^\vee \wedge \dots e_{i_n}^\vee$ to $(e_{i_1} \wedge \dots \wedge e_{i_n})^\vee$ for any $i_1 < \dots < i_n$, from which we immediately see that this is an isomorphism.

This proves the desired statement in the case of $G = \mathbb{Z}$. Since the cohomology groups are all free $\mathbb{Z}$-modules, the general statement follows by universal coefficients. □

The following lemma makes this explicit in terms of inhomogeneous cochains for $n \leq 2$:

Lemma A.2. *Let $C^n := \mathrm{Map}(M^n, G)$ be the inhomogeneous cochains and write Z^n and B^n for the cocycles and coboundaries, so that $H^n(M,G) = Z^n/B^{n-1}$. Then:*

1. *For $n = 1$, we have $B^0 = 0$ and $Z^1 = \mathrm{Hom}(M,G) = H^1(M,G)$.*
2. *For $n = 2$, we have $\mathrm{Hom}(M \times M, G) \subseteq Z^2$ and $\mathrm{Hom}(M \times M, G)^{\mathrm{sym}} \subseteq B^1$. The cohomology can then explicitly described by the short exact sequence*

$$0 \to \mathrm{Hom}(M \otimes M, G)^{\mathrm{sym}} \to \mathrm{Hom}(M \otimes M, G) \xrightarrow{\Delta} \mathrm{Hom}(\wedge^2 M, G) = H^2(M,G) \to 0$$

where Δ is given by sending $f : M \otimes M \to \mathbb{Z}$ to $m_1 \wedge m_2 \mapsto f(m_1,m_2) - f(m_2,m_1)$.

Proof. The case of $n = 1$ is clear from the transition map

$$\partial^1 : C^1(M, G) \to C^2(M, G), \quad \partial^1(f)(m_1, m_2) = f(m_1) + f(m_2) - f(m_1 + m_2).$$

For the case of $n = 2$, it suffices to consider the case $G = \mathbb{Z}$; the general case follows by applying $- \otimes_{\mathbb{Z}} G$ since all appearing modules are free.

We first note that the described short exact sequence is indeed exact: Left-exactness is clear. To see right-exactness, we identify the middle term with $M_n(\mathbb{Z})$, the first term with symmetric matrices, and the third term with matrices which are anti-symmetric. It is then clear on elementary matrices that Δ is surjective.

That $\mathrm{Hom}(M \otimes M, \mathbb{Z}) \subseteq Z^2$ follows from $\partial^2 : C^2(M, \mathbb{Z}) \to C^3(M, \mathbb{Z})$ being

$$\partial^2(f)(m_1, m_2, m_3) = f(m_2, m_3) - f(m_1 + m_2, m_3) + f(m_1, m_2 + m_3) - f(m_1, m_2),$$

which vanishes whenever $f : M \times M \to \mathbb{Z}$ is bilinear. Since the cup product sends $\mathrm{Hom}(M, \mathbb{Z}) \subseteq C^1(M, \mathbb{Z})$ into $\mathrm{Hom}(M \otimes M, \mathbb{Z}) \subseteq C^2(M, \mathbb{Z})$, it follows from Lemma A.1 that the image of $\mathrm{Hom}(M \otimes M, \mathbb{Z})$ generates $H^2(M, \mathbb{Z})$.

It therefore remains to prove that $f : M \times M \to \mathbb{Z}$ is in B^1 if and only if f is symmetric. Since we will want to use this calculation later in a slightly different setup, we make this a lemma on its own. □

The following lemma finishes the proof.

Lemma A.3. *Let G be any abelian group. Then a bilinear function $f : M \times M \to G$ is symmetric if and only if there is a map $r : M \to G$ such that for all $m_1, m_2 \in M$:*

$$-r(m_1 + m_2) + r(m_1) + r(m_2) = f(m_1, m_2).$$

Proof. It is clear that existence of r implies symmetry. To see the converse, let us choose a basis $e_1, \ldots, e_r$ of M. We then define a function $r : M \to G$

$$\begin{aligned} r(m) := & \quad -\tfrac{1}{2}\big(f(m, m) - \sum_{i=1}^{r} e_i^\vee(m) f(e_i, e_i)\big). \\ = & \quad \sum_{1 \leq i < j \leq r} a_i a_j f(e_i, e_j) + \sum_{i=1}^{r} \frac{a_i(a_i - 1)}{2} f(e_i, e_i). \end{aligned}$$

Since the second summand in the first line is linear, we then have

$$r(m_1) + r(m_2) - r(m_1 + m_2) = \tfrac{1}{2} f(m_1 + m_2, m_1 + m_2) - \tfrac{1}{2} f(m_1, m_1) - \tfrac{1}{2} f(m_2, m_2) = f(m_1, m_2),$$

where we have used in the second step that f is bilinear and symmetric. □

Lemma A.4. *Let $N \subseteq M$ be a split submodule. Then the image of* $\mathrm{Hom}(M \otimes (M/N), G)$ *under Δ from Lemma A.2 are precisely those $\wedge^2 M \to G$ that vanish on $\wedge^2 N \subseteq \wedge^2 M$.*

Proof. This follows from the natural isomorphism

$$\wedge^2(M/N) \times (N \otimes M/N) \xrightarrow{\sim} \wedge^2 M / \wedge^2 N.$$

Namely, choose any splitting $M = N \oplus M/N$, we have

$$\mathrm{Hom}(M \otimes (M/N), G) = \mathrm{Hom}(M/N \otimes M/N, G) \oplus \mathrm{Hom}(N \otimes M/N, G).$$

Now sending $f \mapsto (m_1, m_2 \mapsto f(m_1, m_2) - f(m_2, m_1))$ sends the first summand onto $\mathrm{Hom}(\wedge^2(M/N), G)$ by Lemma A.2. □

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